

# Ergodic and Mixing Quantum Channels in Finite Dimensions

Daniel Burgarth, Giulio Chiribella, Vittorio Giovannetti, Paolo Perinotti, and Kazuya Yuasa

**Abstract**—The paper provides a systematic characterization of quantum ergodic and mixing channels in finite dimensions and a discussion of their structural properties. Notably, we show that ergodicity is stable under randomizations, namely that every random mixture of an ergodic channel with a generic channel is still ergodic. In addition, we prove several conditions under which ergodicity can be promoted to the stronger property of mixing. Finally, exploiting a suitable correspondence between quantum channels and generators of quantum dynamical semigroups, we extend our results to the realm of continuous-time quantum evolutions, providing a characterization of ergodic Lindblad generators and showing that they are dense in the set of all possible generators.

## I. INTRODUCTION

IN the study of signal processing, a stochastic process is said to be ergodic if its statistical properties can be deduced from a single, sufficiently long realization of the process. Ergodicity plays a fundamental role in the study of the irreversible dynamics associated with the relaxation to the thermal equilibrium [1]–[4]. In quantum mechanics the evolution of an open system, interacting with an external (initially uncorrelated) environment, is fully characterized in terms of special linear maps  $\mathcal{M}$  (known as *quantum channels*) operating on the space of density matrices  $\rho$  of the system of interest, under certain structural constraints (quantum channels and all their extensions should preserve the positivity and the trace of the operators on which they act upon, see e.g. [5]–[8]). In this framework a convenient formulation of the property of ergodicity is the following: a channel  $\mathcal{M}$  is ergodic if and only if it admits a unique fixed point in the space of density matrices, that is, if there is only one density matrix  $\rho_*$  that

is unaltered by the action of  $\mathcal{M}$  [1], [9]–[11]. The rationale behind such formulation is clear when we consider the discrete trajectories associated with the evolution of a generic input state  $\rho$ , evolving under iterated applications of the transformation  $\mathcal{M}$ : in this case, the mean value of a generic observable  $A$ , averaged over the trajectories, converges asymptotically to the expectation value  $\text{Tr}[A\rho_*]$  of the observable on the fixed point. A related, but stronger property is *mixing*, namely the property that after a sufficiently large number of repeated applications of the channel, the output density matrix converges to the fixed point.

Beyond the study of relaxation processes, ergodicity and mixing have found important applications in several fields of quantum information theory. Most notably in quantum control [12]–[17], quantum estimation [18], quantum communication [19], [20], and in the study of efficient tensorial representation of critical many-body quantum systems [21]–[26]. A complete characterization of these maps appears hence to be mandatory. Here we contribute to this goal by providing a systematic characterization of the structural properties of ergodic and mixing channels on finite-dimensional quantum systems. In particular, we discuss a fundamental property that to our knowledge has never been clarified before, namely the fact that ergodicity is stable under randomizations. Precisely, we show that a convex combination of an ergodic channel with a generic (not necessarily ergodic) channel yields a new transformation which is always ergodic. In addition, we show that the fixed point of the new ergodic map is related with the fixed point of the original ergodic channel via a convex combination. These results extend to the case of ergodic channels a property that was previously known to hold for the restricted subset of mixing channels [13], [14], [27]. In addition, we provide a series of conditions under which ergodicity can be upgraded to the stronger property of mixing: for example, we show that, rather counterintuitively, convex combinations of ergodic channels with the identity channel are always mixing. Another remarkable property is that a random-unitary channel  $\mathcal{M}$  (a channel that is a convex combination of unitaries) in dimension  $d$  is mixing if and only if the channel  $\mathcal{M}^d$  obtained by applying  $\mathcal{M}$  on the system  $d$  times is ergodic. Finally, we extend our characterization of ergodicity to the realm of continuous-time quantum evolutions, for which the properties of ergodicity and mixing are actually equivalent. In this context, we show that the ergodicity of a quantum dynamical semigroup is equivalent to the ergodicity of a suitable quantum channel. Using this result, we show that the generators of ergodic quantum semigroups are stable under randomizations, and, therefore, are dense in the set of all possible generators.

We acknowledge discussions with David Reeb and Kai Schwiager. This work is partially supported by the Joint Italian-Japanese Laboratory on “Quantum Technologies: Information, Communication and Computation” of the Italian Ministry of Foreign Affairs. GC acknowledges support by the National Basic Research Program of China (973) 2011CBA00300 (2011CBA00301) and the hospitality of the Institute of Theoretical Computer Science and Communications, Chinese University of Hong Kong. VG acknowledges support from MIUR through FIRB-IDEAS Project No. RBID08B3FM. KY is supported by the Grant-in-Aid for Young Scientists (B) (No. 21740294) from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

D. Burgarth is with Institute of Mathematics and Physics, Aberystwyth University, Physical Sciences Building, Penglains Campus, SY23 3BZ Aberystwyth, United Kingdom.

G. Chiribella is with Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing 100084, China.

V. Giovannetti is with NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, piazza dei Cavalieri 7, I-56126 Pisa, Italy.

P. Perinotti is with Dipartimento di Fisica, Università degli Studi di Pavia and INFN sezione di Pavia, via Bassi 6, I-27100 Pavia, Italy.

K. Yuasa is with Department of Physics, Waseda University, Tokyo 169-8555, Japan.

The results of this paper are presented in a systematic way, starting from the case of general channels and then specializing to particular classes of channels (such as random-unitary channels). In our presentation we also include for completeness some alternative proofs of existing results, which come out quite naturally in our approach. The paper is organized as follows. We start in Section II by reviewing some basic definitions and properties. In Section III we provide a characterization of ergodicity in terms of the invariant subspaces of the channel. Here we also discuss how ergodicity and mixing of the adjoint map are connected with those of the original channel. The convexity properties of ergodic channel are analyzed in Section IV. Section V instead focuses on the relation between ergodicity and mixing. In particular here we show that a generic convex combination of the identity map with an ergodic channel is mixing. In Section VI we focus on the special case of channels which admit faithful fixed points providing a characterization of the peripheral spectrum of the maps and introducing a necessary and sufficient condition which ergodic maps have to fulfill in order to be mixing. Section VII discusses instead the ergodic/mixing properties of unital maps. Section VIII draws some conclusions on dynamical semigroups. Final remarks are presented in Section IX. The paper also contains an Appendix dedicated to the more technical aspects of the proofs.

## II. DEFINITIONS AND BASIC PROPERTIES

Consider a quantum system with associated Hilbert space  $\mathcal{H}$  of finite dimension  $d < \infty$ . In what follows we will use the symbols  $\mathcal{B}(\mathcal{H})$  and  $\mathfrak{S}(\mathcal{H}) [\subset \mathcal{B}(\mathcal{H})]$  to represent the set of linear operators on  $\mathcal{H}$  and the set of the density matrices, respectively. A quantum channel operating on the system is then defined as a linear mapping  $\mathcal{M} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  which is completely positive and trace-preserving (CPTP in brief). While referring the reader to [5]–[8] for an exhaustive review on the subject, we find it useful to recall a few basic properties of CPTP maps, which will be exploited in the following.

- 1) A linear map  $\mathcal{M} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is completely positive if and only if it is possible to identify a collection of Kraus operators  $\{M_i\}_{i \in \mathcal{X}}$  which, for all  $A \in \mathcal{B}(\mathcal{H})$ , allows us to write

$$\mathcal{M}(A) = \sum_{i \in \mathcal{X}} M_i A M_i^\dagger. \quad (1)$$

Furthermore  $\mathcal{M}$  is trace-preserving if and only if the operators  $M_i$  satisfy the normalization condition

$$\sum_{i \in \mathcal{X}} M_i^\dagger M_i = I. \quad (2)$$

- 2) The set  $\mathfrak{C}(\mathcal{H})$  formed by the quantum channels on the system is closed under convex combination and multiplication, i.e. given  $\mathcal{M}_1, \mathcal{M}_2 \in \mathfrak{C}(\mathcal{H})$  and  $p \in [0, 1]$ , the transformations defined by the mappings

$$A \in \mathcal{B}(\mathcal{H}) \rightarrow p\mathcal{M}_1(A) + (1-p)\mathcal{M}_2(A), \quad (3)$$

$$A \in \mathcal{B}(\mathcal{H}) \rightarrow (\mathcal{M}_1 \circ \mathcal{M}_2)(A) := \mathcal{M}_1(\mathcal{M}_2(A)) \quad (4)$$

are also elements of  $\mathfrak{C}(\mathcal{H})$ .

- 3) Any CPTP map  $\mathcal{M}$  is nonexpansive: when  $\mathcal{M}$  is applied to a couple of input states  $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$ , it produces output density matrices  $\mathcal{M}(\rho), \mathcal{M}(\sigma)$  whose relative distance is not greater than the original one, i.e.

$$\|\mathcal{M}(\rho) - \mathcal{M}(\sigma)\|_1 \leq \|\rho - \sigma\|_1, \quad (5)$$

where  $\|A\|_1 := \text{Tr}[\sqrt{A^\dagger A}]$  is the trace norm.

Since  $\mathcal{M}$  is a linear operator defined on a linear space of dimension  $d^2$ , it admits up to  $d^2$  distinct (complex) eigenvalues  $\lambda$  which solve the equation

$$\mathcal{M}(A) = \lambda A, \quad (6)$$

for some  $A \in \mathcal{B}(\mathcal{H})$ ,  $A \neq 0$ . Such eigenvalues can be determined as the zeros of the associated characteristic polynomial, i.e.

$$\text{Poly}^{(\mathcal{M})}(\lambda) = \text{Det}\left(\lambda I - \sum_{i \in \mathcal{X}} M_i \otimes \bar{M}_i\right) = 0, \quad (7)$$

where  $\{M_i\}_{i \in \mathcal{X}}$  is a set of Kraus operators for  $\mathcal{M}$  and  $\bar{M}_i$  is the operator obtained by taking the entry-wise complex conjugate of  $M_i$  with respect to a selected basis of  $\mathcal{H}$ .

- 4) The spectrum of  $\mathcal{M}$  is invariant under complex conjugation: if  $\lambda \in \mathbb{C}$  is an eigenvalue with eigenvector  $A$ , then its complex conjugate  $\bar{\lambda}$  is an eigenvalue with eigenvector given by the adjoint operator  $A^\dagger$ , namely

$$\mathcal{M}(A^\dagger) = \bar{\lambda} A^\dagger, \quad (8)$$

with  $\bar{\lambda}$  being the complex conjugate of  $\lambda$  and  $A^\dagger$  being the adjoint of  $A$ . This property holds not only for quantum channels, but also for all linear maps that are Hermitian-preserving (that is, they send Hermitian operators to Hermitian operators).

- 5) The eigenvalues of a CPTP map  $\mathcal{M}$  are confined in the unit circle on the complex plane. In other words, if there exists a non-zero  $A \in \mathcal{B}(\mathcal{H})$  such that (6) holds, then we must have  $|\lambda| \leq 1$  [this property is indeed a direct consequence of 3)].

The eigenvalues of  $\mathcal{M}$  which lie at the boundary of the permitted region, i.e. which have unit modulus  $|\lambda| = 1$ , are called *peripheral*. Of particular interest for us is the unit eigenvalue  $\lambda = 1$ : the associated eigenvectors  $A \in \mathcal{B}(\mathcal{H})$  are called *fixed points* of  $\mathcal{M}$  to stress the fact that they are left unchanged by the action of the map  $\mathcal{M}$ :

- 5) Every CPTP map admit *at least one* fixed point state, i.e. a solution of (6) for  $\lambda = 1$ , which belongs to the set  $\mathfrak{S}(\mathcal{H})$  of the density matrices of the system.

As a matter of fact, a generic quantum channel possesses more than just one density matrix that fulfills the requirement 5) (for instance unitary transformations admit infinitely many fixed point states). In the rest of the paper however we will focus on the special subset of CPTP maps which have exactly a single element of  $\mathfrak{S}(\mathcal{H})$  that is stable under the transformation:

**Definition 1.** A CPTP map  $\mathcal{M}$  is said to be *ergodic* if there exists a unique state  $\rho_* \in \mathfrak{S}(\mathcal{H})$  which is left unchanged by

the channel  $\mathcal{M}$ , i.e. which solves (6) for  $\lambda = 1$ . We introduce the symbol  $\mathfrak{C}_E(\mathcal{H})$  to represent the subset containing all the ergodic elements of  $\mathfrak{C}(\mathcal{H})$ .

As it will be explicitly shown in the next section (see Corollary 2), the ergodicity is strong enough to guarantee that  $\rho_*$  is not only the unique  $\lambda = 1$  solution of (6) on  $\mathfrak{S}(\mathcal{H})$  but also (up to a multiplicative factor) the only solution for the same problem in the larger set of the operators  $\mathcal{B}(\mathcal{H})$ . An equivalent (and possibly more intuitive) way to define ergodicity can be obtained by posing a constraint on the effective *discrete-time* evolution generated by the repetitive applications of  $\mathcal{M}$ . Specifically, given  $\rho \in \mathfrak{S}(\mathcal{H})$  a generic input state of the system, consider the series

$$\Sigma_N^{(\mathcal{M})}(\rho) = \frac{1}{N+1} \sum_{n=0}^N \mathcal{M}^n(\rho), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}), \quad (9)$$

where  $\mathcal{M}^0 = \mathcal{I}$  stands for the identity superoperator, while for  $n \geq 1$ ,  $\mathcal{M}^n$  is a CPTP map associated with  $n$  recursive applications of  $\mathcal{M}$ , i.e.

$$\mathcal{M}^n := \underbrace{\mathcal{M} \circ \dots \circ \mathcal{M}}_{n \text{ times}}. \quad (10)$$

Equation (9) describes the average state associated with the first  $N+1$  steps of the discrete trajectory of  $\mathfrak{S}(\mathcal{H})$ , defined by the density matrices  $\rho, \mathcal{M}(\rho), \mathcal{M}^2(\rho), \dots, \mathcal{M}^N(\rho)$ , obtained by applying  $\mathcal{M}$  to  $\rho$  recursively. It then turns out that  $\mathcal{M}$  is in  $\mathfrak{C}_E(\mathcal{H})$  and has a unique fixed point state  $\rho_*$ , if and only if  $\Sigma_N^{(\mathcal{M})}(\rho)$  converges in trace norm to  $\rho_*$ , i.e.

$$\lim_{N \rightarrow \infty} \|\Sigma_N^{(\mathcal{M})}(\rho) - \rho_*\|_1 = 0 \quad (11)$$

(see for example [9] for an explicit proof of this fact). Accordingly, this implies that for ergodic channels the average of the expectation values  $a_n = \text{Tr}[A\mathcal{M}^n(\rho)]$  of any bounded operator  $A$ , evaluated along the trajectory  $\rho, \mathcal{M}(\rho), \mathcal{M}^2(\rho), \dots, \mathcal{M}^N(\rho)$ , converges to the fixed point value  $a_* = \text{Tr}[A\rho_*]$ , i.e.

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N a_n = a_*. \quad (12)$$

A proper subset of  $\mathfrak{C}_E(\mathcal{H})$  is constituted by *mixing/relaxing* maps [9]:

**Definition 2.** A quantum channel  $\mathcal{M}$  is said to be *mixing* if  $\exists! \rho_* \in \mathfrak{S}(\mathcal{H})$  such that

$$\lim_{n \rightarrow \infty} \|\mathcal{M}^n(\rho) - \rho_*\|_1 = 0, \quad \forall \rho \in \mathfrak{S}(\mathcal{H}). \quad (13)$$

We call the special state  $\rho_*$  the *fixed point state* of  $\mathcal{M}$  and introduce the symbol  $\mathfrak{C}_M(\mathcal{H})$  to represent the subset containing all the *mixing elements* of  $\mathfrak{C}(\mathcal{H})$ .

As anticipated all mixing maps are ergodic (with their fixed point states provided by the stable density matrices of the mixing channels), i.e.  $\mathfrak{C}_M(\mathcal{H}) \subset \mathfrak{C}_E(\mathcal{H})$ . Notably however the opposite is not true [9]: for instance the qubit channel

$$\mathcal{M}(\rho) = \langle 1|\rho|1\rangle|0\rangle\langle 0| + \langle 0|\rho|0\rangle|1\rangle\langle 1|, \quad (14)$$

is ergodic with fixed point state  $\rho_* = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ , but it does not fulfill the mixing condition (13) [in fact,  $\mathcal{M}^n(|0\rangle\langle 0|)$  keeps oscillating between  $|0\rangle\langle 0|$  and  $|1\rangle\langle 1|$ ]. Interestingly in the case of continuous time Markovian evolution, mixing and ergodicity are equivalent (see Section VIII), which means that the above channel cannot be obtained as the result of a Markovian time evolution (cf. [28]).

The mixing property of a channel  $\mathcal{M}$  can be described in terms of its spectral properties. Indeed a necessary and sufficient condition for mixing is the fact that (up to a multiplicative factor) the fixed point state  $\rho_*$  of  $\mathcal{M}$  is the unique peripheral eigenvector of  $\mathcal{M}$ . More specifically a channel is mixing if and only if it is ergodic *and* no solutions exists in  $\mathcal{B}(\mathcal{H})$  for the eigenvalue equation (6) with both  $|\lambda| = 1$  and  $\lambda \neq 1$ .

**Remark 1.** Requiring  $\lambda = 1$  to be the only peripheral eigenvalue of the channel is not sufficient to enforce the mixing property (or even ergodicity). As a counterexample consider for instance the case of the identity channel  $\mathcal{I}$ .

### III. CHARACTERIZATION OF ERGODICITY IN TERMS OF INVARIANT SUBSPACES

In this section we present a characterization of the ergodicity of a channel  $\mathcal{M}$  in terms of the subspaces that are left invariant by its action.

In particular, we start in Section III-A by presenting a characterization of the fixed point subspace showing that it is spanned by positive operators. Then we link the notion of ergodicity to the properties of invariant subspaces of the channel (Section III-B) and of its adjoint map (Section III-C).

#### A. The linear Space Spanned by Fixed Points

The linear subspace of  $\mathcal{B}(\mathcal{H})$  generated by the fixed points of a channel  $\mathcal{M}$  can be shown to be spanned by positive operators. This fact is a consequence of Proposition 6.8 of [8] which we report here for the sake of completeness:

**Lemma 1.** Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a quantum channel, and let  $A \in \mathcal{B}(\mathcal{H})$  be a fixed point of  $\mathcal{M}$ , written as  $A = (X_+ - X_-) + i(Y_+ - Y_-)$ , where  $X_+, X_-, Y_+, Y_- \in \mathcal{B}_+(\mathcal{H})$  are positive operators such that  $X_+X_- = X_-X_+ = Y_+Y_- = Y_-Y_+ = 0$ . Then, also  $X_+, X_-, Y_+, Y_-$  are fixed points of  $\mathcal{M}$ .

*Proof:* The proof is provided in the Appendix. ■

From this it immediately follows that non-ergodic channels always admit at least two fixed point states that are “not overlapping”:

**Corollary 1.** Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a quantum channel. If  $\mathcal{M}$  has two distinct fixed point states, then  $\mathcal{M}$  has also two fixed point states that have orthogonal supports.

*Proof:* Suppose that two distinct density matrices  $\rho_0$  and  $\rho_1$  are fixed point states of  $\mathcal{M}$ . Then, the traceless operator  $\Delta := \rho_0 - \rho_1 \neq 0$  is also a fixed point of the channel. Denoting by  $\Delta_+ \geq 0$  and  $\Delta_- \geq 0$  the positive and negative parts of



$\Delta$ , respectively, by Lemma 1 we have that  $\Delta_+$  and  $\Delta_-$  are both fixed points, and so are the states  $\rho_{\pm} := \Delta_{\pm}/\text{Tr}[\Delta_{\pm}]$ . Clearly,  $\rho_+$  and  $\rho_-$  have orthogonal supports. ■

Another consequence of Lemma 1 is that all fixed points of an ergodic channel must coincide up to a proportionality constant:

**Corollary 2.** *A CPTP map  $\mathcal{M}$  is ergodic with fixed point state  $\rho_*$ , if and only if all the solutions in  $\mathcal{B}(\mathcal{H})$  of the eigenvalue equation*

$$\mathcal{M}(A) = A \quad (15)$$

can be expressed as

$$A = \rho_* \text{Tr}[A]. \quad (16)$$

*Proof:* If all the solutions of (15) can be expressed as (16), then the map  $\mathcal{M}$  is clearly ergodic, the converse instead follows by contradiction from Lemma 1. ■

It is worth noticing that an alternative proof of the Corollary 2 can also be obtained from Lemma 6 of [9], which states that if  $A$  is a peripheral eigenvector of a channel  $\mathcal{M}$  then  $|A| := \sqrt{A^\dagger A}$  must be a fixed point of  $\mathcal{M}$  (see the Appendix for details).

### B. Fixed Points and Invariant Subspaces

Here we link the ergodicity property of a channel to the structure of its invariant subspaces, that is, of those subspaces of  $\mathcal{H}$  that are left invariant by the action of the Kraus operators of the channel.

**Definition 3.** *We say that a subspace  $S \subseteq \mathcal{H}$  is invariant for a completely positive (not necessarily trace-preserving) channel  $\mathcal{M}$  if and only if for every Kraus representation of the channel  $\mathcal{M}(\rho) = \sum_{i \in \mathcal{X}} M_i \rho M_i^\dagger$  we have  $M_i|\varphi\rangle \in S$  for every  $|\varphi\rangle \in S$  and for every  $i \in \mathcal{X}$ .*

It is worth reminding that the condition that  $S$  is an invariant subspace under  $\mathcal{M}$  can be equivalently expressed by the following properties:

- a)  $M_i P = P M_i P$  for every  $i \in \mathcal{X}$ , where  $P$  is the projector on  $S$ ;
- b)  $\mathcal{M}(P) = P \mathcal{M}(P) P$ ;
- c)  $\text{Supp}[\mathcal{M}(\rho)] \subseteq S$  for every state  $\rho \in \mathfrak{S}(\mathcal{H})$  with  $\text{Supp}(\rho) \subseteq S$  [here  $\text{Supp}(\rho)$  stands for the support of the state  $\rho$ ].

The invariant subspaces of a channel are related to its fixed point states in the following way:

**Lemma 2.** *If  $\rho \in \mathfrak{S}(\mathcal{H})$  is a fixed point state for  $\mathcal{M}$ , then the support of  $\rho$  is an invariant subspace. Moreover, if  $S \subseteq \mathcal{H}$  is an invariant subspace for  $\mathcal{M}$ , then there exists a fixed point state  $\rho_S \in \mathfrak{S}(\mathcal{H})$  with  $\text{Supp}(\rho_S) \subseteq S$ .*

*Proof:* From Lemma 8 of the Appendix it follows that for every unit vector  $|\varphi\rangle \in \text{Supp}(\rho)$  there exists a positive probability  $p > 0$  and a state  $\sigma$  such that  $\rho = p|\varphi\rangle\langle\varphi| + (1 -$

$p)\sigma$ . Furthermore since  $\mathcal{M}(\rho) = p\mathcal{M}(|\varphi\rangle\langle\varphi|) + (1-p)\mathcal{M}(\sigma)$ , the same Lemma implies that we must also have

$$\text{Supp}[\mathcal{M}(|\varphi\rangle\langle\varphi|)] \subseteq \text{Supp}[\mathcal{M}(\rho)]. \quad (17)$$

Consider then the case in which  $\rho$  is a fixed point state for  $\mathcal{M}$ , i.e.  $\mathcal{M}(\rho) = \rho$ . Equation (17) then implies that  $\text{Supp}[\mathcal{M}(|\varphi\rangle\langle\varphi|)] \subseteq \text{Supp}(\rho)$  for every  $|\varphi\rangle \in \text{Supp}(\rho)$ , namely that  $\text{Supp}(\rho)$  is an invariant subspace. Conversely, let  $S$  be an invariant subspace for  $\mathcal{M}$ . Then the restriction of  $\mathcal{M}$  to  $S$  is a channel in  $\mathfrak{C}(S)$ , and, as such, has a fixed point  $\rho_S \geq 0$  with  $\text{Supp}(\rho_S) \subseteq S$ . ■

Using the relation between fixed point states and invariant subspaces we can obtain a first characterization of ergodicity in terms of the invariant subspaces of the channel:

**Theorem 1** (Characterization of ergodicity in terms of invariant subspaces). *For a quantum channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$ , the followings are equivalent:*

- 1)  $\mathcal{M}$  is ergodic;
- 2)  $\mathcal{M}$  does not have two invariant subspaces  $S_1 \neq \{0\}$  and  $S_2 \neq \{0\}$  such that  $S_1 \cap S_2 = \{0\}$ ;
- 3)  $\mathcal{M}$  has a minimal invariant subspace, that is, a subspace  $S \neq \{0\}$  such that  $S \subseteq S'$  for every invariant subspace  $S' \neq \{0\}$  (remark: the minimal invariant subspace coincides with the support of the fixed point state).

*Proof:* 1)  $\Rightarrow$  2) If  $\mathcal{M}$  has two non-intersecting invariant subspaces  $S_1$  and  $S_2$ , then, by Lemma 2 it has two distinct fixed points  $\rho_{S_1}$  and  $\rho_{S_2}$ , respectively. Hence,  $\mathcal{M}$  is not ergodic. 2)  $\Rightarrow$  1) If  $\mathcal{M}$  is not ergodic, then it has two distinct invariant states. By Corollary 1, this implies that  $\mathcal{M}$  has two orthogonal invariant states,  $\rho_+$  and  $\rho_-$ . Hence  $\text{Supp}(\rho_+)$  and  $\text{Supp}(\rho_-)$  are two orthogonal invariant subspaces, and, in particular  $\text{Supp}(\rho_+) \cap \text{Supp}(\rho_-) = \{0\}$ . 1)  $\Rightarrow$  3) If  $\mathcal{M}$  is ergodic with fixed point  $\rho_*$ , then  $\text{Supp}(\rho_*)$  is a minimal invariant subspace. Indeed, for every invariant subspace  $S$  there is an invariant state  $\rho_S$  with  $\text{Supp}(\rho_S) \subseteq S$  (Lemma 2). Now, since  $\mathcal{M}$  is ergodic there is only one invariant state, i.e.  $\rho_S \equiv \rho_*$ . Hence,  $\text{Supp}(\rho_*) \subseteq S$ . 3)  $\Rightarrow$  2) If  $\mathcal{M}$  has two non-intersecting invariant subspaces  $S_1$  and  $S_2$ , then it cannot have a minimal invariant subspace  $S$ , because in that case we should have  $S \subseteq S_1 \cap S_2 = \{0\}$ . ■

As an example consider the case of the erasure channel which maps every state into a given selected state  $\rho_0$  according to the transformation

$$\mathcal{M}(A) = \rho_0 \text{Tr}[A] \quad \forall A \in \mathcal{B}(\mathcal{H}). \quad (18)$$

This map is clearly ergodic with the fixed point state being  $\rho_0$ . We can then easily verify that in agreement with Theorem 1 any invariant subspace  $S \neq \{0\}$  of  $\mathcal{M}$  must necessarily contain the support of  $\rho_0$ . Indeed from condition b) below Definition 3 it follows that  $S$  is invariant under  $\mathcal{M}$  when given  $P$  the projector on  $S$  we have

$$\text{Tr}[P]\rho_0 = \text{Tr}[P]P\rho_0P, \quad (19)$$

but since  $\text{Tr}[P] \neq 0$ , this can only be true if  $\text{Supp}(\rho_0) \subseteq S$ .

Theorem 1 provides a straightforward characterization of ergodicity for random-unitary qubit channels:

**Corollary 3** (Ergodic random-unitary qubit channels). *A random-unitary qubit channel  $\mathcal{M} = \sum_{i \in \mathcal{X}} p_i \mathcal{U}_i$ , with  $\mathcal{U}_i(\rho) = U_i \rho U_i^\dagger$  being unitary transformations and  $p_i > 0$  positive probabilities, is ergodic if and only if within the set  $\{U_i\}_{i \in \mathcal{X}}$  there exist at least two elements which do not commute.*

*Proof:* If all the unitaries  $U_i$  commute, they can be jointly diagonalized, and every joint eigenvector is a fixed point, implying that the channel is not ergodic. Conversely, if at least two unitaries of the set do not commute, the only invariant subspace  $S \neq 0$  is  $S = \mathcal{H}$ . Hence,  $S$  is clearly minimal and Theorem 1 guarantees that  $\mathcal{M}$  is ergodic. ■

**Remark 2.** *We remind that in the case of qubit channels the set of random-unitaries coincides with the set of unital maps (i.e. with the set of CPTP maps which admit the identity operator as fixed point) [29]. Corollary 3 hence provides a complete characterization of ergodicity for qubit unital maps. A (partial) generalization of this result to the case of unital maps operating on higher dimensional Hilbert spaces is given in Section VII.*

### C. Ergodicity in Terms of the Adjoint Map

Here we give an equivalent condition for the ergodicity of a quantum channel  $\mathcal{M}$  in terms the adjoint map  $\mathcal{M}^\dagger$ , defined as follows:

**Definition 4.** *Given a linear map  $\mathcal{M} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ , the adjoint of  $\mathcal{M}$  is the linear map  $\mathcal{M}^\dagger : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  uniquely defined by the relation*

$$\langle A, \mathcal{M}(B) \rangle = \langle \mathcal{M}^\dagger(A), B \rangle, \quad (20)$$

where  $\langle A, B \rangle := \text{Tr}[A^\dagger B]$  is the Hilbert-Schmidt product.

Going from a quantum channel to its adjoint is the same as going from the Schrödinger picture to the Heisenberg picture: if  $\mathcal{M}$  represents the evolution of the states, then  $\mathcal{M}^\dagger$  represents the evolution of the observables. It is well known that  $\mathcal{M}$  is (completely) positive if and only if  $\mathcal{M}^\dagger$  is (completely) positive and that  $\mathcal{M}$  is trace-preserving if and only if  $\mathcal{M}^\dagger$  is identity-preserving, i.e.

$$\mathcal{M}^\dagger(I) = I. \quad (21)$$

When  $\mathcal{M}$  is completely positive, a Kraus representation for  $\mathcal{M}^\dagger$  can be obtained by taking the adjoint of the Kraus operators of  $\mathcal{M}$ : if  $\mathcal{M}(\rho) = \sum_{i \in \mathcal{X}} M_i \rho M_i^\dagger$ , then

$$\mathcal{M}^\dagger(A) = \sum_{i \in \mathcal{X}} M_i^\dagger A M_i, \quad \forall A \in \mathcal{B}(\mathcal{H}). \quad (22)$$

As a consequence, the spectra of the two maps are identical, i.e. they share the same eigenvalues.

Indeed, (22) implies that the characteristic polynomials (7) of the two maps coincide up to complex conjugation, i.e.

$$\text{Poly}^{(\mathcal{M}^\dagger)}(\lambda) = 0 \iff \text{Poly}^{(\mathcal{M})}(\bar{\lambda}) = 0. \quad (23)$$

Since the spectrum of a completely positive map is invariant under complex conjugation [cf. property 4) in Section II], this proves that  $\mathcal{M}$  and  $\mathcal{M}^\dagger$  have the same spectrum.

In view of the above result it makes sense to extend the definition of ergodicity and mixing also for the adjoints of CPTP channels:

**Definition 5.** *Given a CPTP map  $\mathcal{M}$  we say that its adjoint  $\mathcal{M}^\dagger$  is ergodic when (up to a proportionality constant) it admits only the operator  $I$  as the fixed point and mixing if furthermore it does not possess other peripheral eigenvalues.*

In other words,  $\mathcal{M}^\dagger$  is defined to be ergodic if  $\mathcal{M}$  has only trivial constants of motion: an observable  $A \in \mathcal{B}(\mathcal{H})$  is a constant of motion for the channel  $\mathcal{M}$  if for every state  $\rho \in \mathfrak{S}(\mathcal{H})$  we have  $\text{Tr}[A \mathcal{M}(\rho)] = \text{Tr}[A \rho]$ , or, equivalently, if  $\mathcal{M}^\dagger(A) = A$ . Saying that  $\mathcal{M}^\dagger$  is ergodic amounts to saying that the only independent constant of motion of the trace of the density matrix.

With the above definitions one can prove that a CPTP map  $\mathcal{M}$  is ergodic (or mixing) if and only if its adjoint  $\mathcal{M}^\dagger$  is ergodic (or mixing):

**Theorem 2** (Ergodicity and constants of motion). *For a channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  the followings are equivalent:*

- 1)  $\mathcal{M}$  is ergodic;
- 2) all the constants of motion are multiples of the identity;
- 3) the adjoint map  $\mathcal{M}^\dagger$  is ergodic.

*Proof:* 1)  $\Rightarrow$  2) Consider the quantum channels  $\Sigma_N^{(\mathcal{M})}$  defined in (9). As discussed in Section II, if  $\mathcal{M}$  is ergodic with fixed point  $\rho_*$ , then for each bounded operator  $A$  and for all density matrices  $\rho$  we must have

$$\lim_{N \rightarrow \infty} \langle A, \Sigma_N^{(\mathcal{M})}(\rho) \rangle = \langle A, \rho_* \rangle. \quad (24)$$

By linearity the adjoint channel of  $\Sigma_N^{(\mathcal{M})}$  is given by  $\Sigma_N^{(\mathcal{M}^\dagger)}$ . Therefore if  $A$  is a constant of motion for  $\mathcal{M}$ , then it is also a constant of motion for  $\Sigma_N^{(\mathcal{M})}$ , i.e.  $\Sigma_N^{(\mathcal{M}^\dagger)}(A) = A$ . In this case (24) can be written as

$$\langle A, \rho_* \rangle = \lim_{N \rightarrow \infty} \langle \Sigma_N^{(\mathcal{M}^\dagger)}(A), \rho \rangle = \langle A, \rho \rangle, \quad (25)$$

which, to be true for all  $\rho$ , implies  $A = \langle A, \rho_* \rangle I$ . 2)  $\Rightarrow$  1) Suppose that all constants of motion of  $\mathcal{M}$  are proportional to the identity. Assume then by contradiction that  $\mathcal{M}$  is not ergodic. By Corollary 1 we know that  $\mathcal{M}$  must have two orthogonal invariant states  $\rho_0$  and  $\rho_1$ . Take then two orthogonal projectors  $P_0 \neq 0$  and  $P_1 \neq 0$  such that  $\langle P_i, \rho_j \rangle = \delta_{ij}$ , and define the operators  $P_{i,\infty} := \lim_{N \rightarrow \infty} \Sigma_N^{(\mathcal{M}^\dagger)}(P_i)$  ( $i = 0, 1$ ). One can easily verify that  $P_{0,\infty}$  and  $P_{1,\infty}$  are constants of motion of  $\mathcal{M}$  [indeed they verify the identities  $\mathcal{M}^\dagger(P_{i,\infty}) = P_{i,\infty}$ ]. However since  $\rho_0$  and  $\rho_1$  are fixed point states of the map, we also have  $\langle P_{i,\infty}, \rho_j \rangle = \lim_{N \rightarrow \infty} \langle P_i, \Sigma_N^{(\mathcal{M})}(\rho_j) \rangle = \langle P_i, \rho_j \rangle = \delta_{ij}$ , which is in contradiction to the fact that  $P_{0,\infty}$  and  $P_{1,\infty}$  should be multiples of the identity. 2)  $\iff$  3) Obvious from the definition. ■

**Corollary 4.** *A CPTP map  $\mathcal{M}$  is mixing if and only if its adjoint channel  $\mathcal{M}^\dagger$  admits the identity operator as unique eigenvector associated with a peripheral eigenvalue.*

*Proof:* Recall that  $\mathcal{M}^\dagger$  and  $\mathcal{M}$  share the same spectrum. Having  $\mathcal{M}$  mixing implies then that  $\lambda = 1$  is the unique peripheral eigenvalue of  $\mathcal{M}^\dagger$ . The first implication then follows from Theorem 2 by recalling that any mixing channel  $\mathcal{M}$  is also ergodic. Conversely, if  $\mathcal{M}^\dagger$  admits the identity operator as unique eigenvector associated with peripheral eigenvalues then by Theorem 2  $\mathcal{M}$  is ergodic and no other peripheral eigenvalues can exist, i.e. it is mixing. ■

We have already noticed that the spectrum of a quantum channel  $\mathcal{M}$  coincides with that of its adjoint  $\mathcal{M}^\dagger$ . Here we strengthen this result by showing that, at least for peripheral eigenvalues, there is a simple relation which connects the associated eigenvectors:

**Lemma 3.** *Let  $\mathcal{M}$  be a (not necessarily ergodic) quantum channel with fixed point state  $\rho_*$ . If  $\omega \in \mathbb{C}$  with  $|\omega| = 1$  is an eigenvalue of  $\mathcal{M}^\dagger$  with eigenvector  $A \in \mathcal{B}(\mathcal{H})$ , then  $\bar{\omega}$  is an eigenvalue of  $\mathcal{M}$  with eigenvector  $\tilde{A} := A\rho_*$ . The converse holds if  $\mathcal{M}$  has a strictly positive fixed point  $\rho_* > 0$ : under this hypothesis, if  $\bar{\omega} \in \mathbb{C}$  with  $|\bar{\omega}| = 1$  is an eigenvalue of  $\mathcal{M}$  with eigenvector  $\tilde{A} \in \mathcal{B}(\mathcal{H})$ , then  $\omega$  is an eigenvalue of  $\mathcal{M}^\dagger$  with eigenvector  $A := \tilde{A}\rho_*^{-1}$ .*

*Proof:* Suppose that  $\mathcal{M}^\dagger(A) = \omega A$ . Then, introducing Kraus operators  $\{M_i\}_{i \in X}$  for  $\mathcal{M}$  we have

$$\begin{aligned} \text{Tr}[A^\dagger A\rho_*] &= \omega \langle \mathcal{M}^\dagger(A), A\rho_* \rangle = \omega \sum_i \text{Tr}[M_i A\rho_* M_i^\dagger A^\dagger] \\ &\leq \sqrt{\sum_i \text{Tr}[M_i A\rho_* A^\dagger M_i^\dagger] \cdot \sum_j \text{Tr}[A M_j \rho_* M_j^\dagger A^\dagger]} \\ &= \sqrt{\text{Tr}[\mathcal{M}(A\rho_* A^\dagger)] \text{Tr}[A \mathcal{M}(\rho_*) A^\dagger]} = \text{Tr}[A^\dagger A\rho_*], \quad (26) \end{aligned}$$

where we have used the fact that  $\rho_*$  is a fixed point state for  $\mathcal{M}$ , the cyclicity of the trace, and the Cauchy-Schwarz inequality. Since the equality can only be obtained when the Cauchy-Schwarz inequality is saturated, we must have  $\omega M_i A\rho_*^{1/2} = A M_i \rho_*^{1/2}$  for all  $i$ . Multiplying by  $\rho_*^{1/2} M_i^\dagger$  on the right and summing over  $i$ , we obtain the identity  $\omega \mathcal{M}(A\rho_*) = \omega \sum_i M_i A\rho_* M_i^\dagger = A \sum_i M_i \rho_* M_i^\dagger = A \mathcal{M}(\rho_*) = A\rho_*$ , which shows that  $\tilde{A} = A\rho_*$  is indeed the eigenvector of  $\mathcal{M}$  belonging to the eigenvalue  $\bar{\omega}$ . Conversely, suppose that  $\mathcal{M}$  has a strictly positive fixed point  $\rho_* > 0$  and that  $\mathcal{M}(\tilde{A}) = \bar{\omega} \tilde{A}$ . Defining  $A = \tilde{A}\rho_*^{-1}$  we have

$$\begin{aligned} \text{Tr}[A^\dagger A\rho_*] &= \omega \langle A, \mathcal{M}(A\rho_*) \rangle = \omega \sum_i \text{Tr}[M_i A\rho_* M_i^\dagger A^\dagger] \\ &\leq \sqrt{\sum_i \text{Tr}[M_i A\rho_* A^\dagger M_i^\dagger] \cdot \sum_j \text{Tr}[A M_j \rho_* M_j^\dagger A^\dagger]} \\ &= \sqrt{\text{Tr}[\mathcal{M}(A\rho_* A^\dagger)] \text{Tr}[A \mathcal{M}(\rho_*) A^\dagger]} = \text{Tr}[A^\dagger A\rho_*]. \quad (27) \end{aligned}$$

Again, to attain the equality in the Cauchy-Schwarz inequality, we must have  $\omega M_i A\rho_*^{1/2} = A M_i \rho_*^{1/2}$  for all  $i$ , or equivalently,  $\omega M_i A = A M_i$  for all  $i$ . Hence, we have

$$\mathcal{M}^\dagger(A) = \sum_i M_i^\dagger A M_i = \omega \sum_i M_i^\dagger M_i A = \omega \mathcal{M}^\dagger(I) A = \omega A. \quad \blacksquare$$

It is worth noticing that, for  $\omega = 1$ , Lemma 3 is trivially verified by the eigenoperator  $I$  of  $\mathcal{M}^\dagger$ .

The ergodicity of a CPTP map  $\mathcal{M}$  can also be characterized in terms of the invariant subspaces of  $\mathcal{M}^\dagger$ . For this purpose, observe that the invariant subspaces of a positive map  $\mathcal{M}$  are related to the invariant subspaces of its adjoint  $\mathcal{M}^\dagger$  in the following way:

**Lemma 4.** *Let  $\mathcal{M}$  be a positive map. Then  $S \subseteq \mathcal{H}$  is an invariant subspace for  $\mathcal{M}$  if and only if  $S^\perp := \{|\varphi\rangle \in \mathcal{H} \mid \langle \varphi | \psi \rangle = 0, \forall |\psi\rangle \in S\}$  is an invariant subspace for  $\mathcal{M}^\dagger$ .*

*Proof:* Let us denote by  $P$  and  $P^\perp$  the projectors on  $S$  and  $S^\perp$ , respectively. For a completely positive map  $\mathcal{M}$  we have that  $S$  is invariant if and only if  $\mathcal{M}(P) = P\mathcal{M}(P)P$  [see the equivalent condition b) below Definition 3]. If this constraint is satisfied, then we have

$$\begin{aligned} \langle \mathcal{M}^\dagger(P^\perp), P \rangle &= \langle P^\perp, \mathcal{M}(P) \rangle \\ &= \langle P^\perp, P\mathcal{M}(P)P \rangle \\ &= \langle PP^\perp P, \mathcal{M}(P) \rangle = 0, \quad (28) \end{aligned}$$

which implies  $P\mathcal{M}^\dagger(P^\perp)P = 0$ . By positivity of  $\mathcal{M}^\dagger(P^\perp)$ , we necessarily have  $\mathcal{M}^\dagger(P^\perp) = P^\perp \mathcal{M}^\dagger(P^\perp) P^\perp$ . Hence,  $S^\perp$  is an invariant subspace for  $\mathcal{M}^\dagger$ . Repeating the same argument, we have that if  $S^\perp$  is invariant for  $\mathcal{M}^\dagger$ , then  $S$  is invariant for  $\mathcal{M}$ . ■

Using this fact, the characterization of Theorem 1 becomes

**Theorem 3** (Characterization of ergodicity in terms of invariant subspace of the adjoint map). *For a quantum channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$ , the followings are equivalent:*

- 1)  $\mathcal{M}$  is ergodic;
- 2)  $\mathcal{M}^\dagger$  does not have two invariant subspaces  $S_1 \neq \mathcal{H}$  and  $S_2 \neq \mathcal{H}$  such that  $\text{Span}(S_1 \cup S_2) = \mathcal{H}$ ;
- 3)  $\mathcal{M}^\dagger$  has a maximal invariant subspace, that is, a subspace  $S \neq \mathcal{H}$  such that  $S \supseteq S'$  for every invariant subspace  $S' \neq \mathcal{H}$  (remark: the maximal invariant subspace coincides with the kernel of the fixed point of  $\mathcal{M}$ ).

Note that in the above Theorem 3 the subspace maximal invariant subspace  $S$  can consist only of the zero vector, if the minimal invariant subspace of  $\mathcal{M}$  is the whole Hilbert space.

#### IV. STABILITY OF ERGODICITY UNDER RANDOMIZATION

It is known that when we prepare a (nontrivial) convex combination of a mixing channel with a generic (not necessarily mixing) quantum channel, the resulting transformation is also mixing [13], [14], [27], implying that  $\mathfrak{C}_M(\mathcal{H})$  constitutes a convex subset, which is dense in  $\mathfrak{C}(\mathcal{H})$ . Here we are going to show that the same property holds for the larger set of ergodic channels  $\mathfrak{C}_E(\mathcal{H})$ . Notice that this fact cannot be established by simple geometric arguments based upon the fact that  $\mathfrak{C}_E(\mathcal{H})$  includes  $\mathfrak{C}_M(\mathcal{H})$ .



**Theorem 4** (Stability of ergodicity under randomization). *Let  $\mathcal{M}$  be an ergodic CPTP map and  $\mathcal{M}'$  an arbitrary (not necessarily ergodic) element of  $\mathfrak{C}(\mathcal{H})$ . Then, for all  $p \in (0, 1]$ , the CPTP map*

$$\mathcal{M}_p := p\mathcal{M} + (1-p)\mathcal{M}', \quad (29)$$

*is also ergodic. Moreover, denoting by  $\rho_*$  and  $\rho_{*,p}$  the fixed point states of  $\mathcal{M}$  and  $\mathcal{M}_p$ , respectively, we have that*

$$\rho_{*,p} = \pi_p \rho_* + (1 - \pi_p) \sigma_p \quad (30)$$

*for some probability  $\pi_p \in (0, 1]$  and for some state  $\sigma_p \in \mathfrak{S}(\mathcal{H})$ .*

*Proof:* Let  $\rho \in \mathfrak{S}(\mathcal{H})$  be a fixed point for  $\mathcal{M}_p$ , so that we have  $p\mathcal{M}(\rho) + (1-p)\mathcal{M}'(\rho) = \rho$ . By Lemma 8 of the Appendix this relation requires  $\text{Supp}[\mathcal{M}(\rho)] \subseteq \text{Supp}(\rho)$ , that is,  $\text{Supp}(\rho)$  is an invariant subspace for  $\mathcal{M}$ . Since  $\mathcal{M}$  is ergodic, by Theorem 1 it has a minimal invariant subspace  $S$ , and hence  $S \subseteq \text{Supp}(\rho)$ . Now, by Theorem 1,  $\mathcal{M}_p$  must be ergodic, because it cannot have two orthogonal invariant subspaces (every invariant subspace of  $\mathcal{M}_p$  must contain  $S$ ). Moreover, recalling that the minimal invariant subspace of  $\mathcal{M}$  is  $S = \text{Supp}(\rho_*)$ , we obtain the relation  $\text{Supp}(\rho_*) \subseteq \text{Supp}(\rho_{*,p})$ , which implies (30) via Lemma 8. ■

We point out that an alternative proof of the fact that  $\mathcal{M}_p$  of (29) is ergodic can be obtained from Corollary 2. This derivation is presented in the Appendix.

## V. FROM ERGODICITY TO MIXING

Once established that a channel  $\mathcal{M}$  is ergodic, one may further ask whether it is also mixing. To answer this question we have to study the peripheral eigenvalues of  $\mathcal{M}$  and to see whether or not  $\omega = 1$  is the only peripheral eigenvalue. A useful observation in this direction is the following results which provide a refinement of Lemma 6 of [9]:

**Lemma 5.** *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a (not necessarily trace-preserving) CP map with Kraus operators  $\{M_i\}_{i \in \mathbb{X}}$ . Then an operator  $A \neq 0$  with polar decomposition  $A = U|A|$  is an eigenvector of  $\mathcal{M}$  belonging to the eigenvalue  $\omega \in \mathbb{C}$  if  $|A|$  is a fixed point of  $\mathcal{M}$  and if the following condition holds:*

$$M_i U |A| = \omega U M_i |A|, \quad \forall i \in \mathbb{X}. \quad (31)$$

*Proof:* The thesis immediately follows by observing that if (31) holds then we have

$$\begin{aligned} \mathcal{M}(A) &= \sum_{i \in \mathbb{X}} M_i U |A| M_i^\dagger \\ &= \omega U \sum_{i \in \mathbb{X}} M_i |A| M_i^\dagger \\ &= \omega U \mathcal{M}(|A|) = \omega U |A| = \omega A, \end{aligned} \quad (32)$$

where in the second to last passage we have used the fact that  $|A|$  is a fixed point of  $\mathcal{M}$ . ■

Quite interestingly, when considering peripheral eigenvalues of a trace-preserving CP map the sufficient condition of Lemma 5 can be transformed into a necessary one:

**Theorem 5** (Peripheral eigenvectors of quantum channels). *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a CPTP map,  $\omega \in \mathbb{C}$  be a complex number on the unit circle ( $|\omega| = 1$ ), and  $A \in \mathfrak{B}(\mathcal{H})$  be an operator with polar decomposition  $A = U|A|$ . The followings are equivalent:*

- 1)  $A$  is an eigenvector of  $\mathcal{M}$  with eigenvalue  $\omega$ ;
- 2)  $|A|$  is a fixed point of  $\mathcal{M}$  and

$$M_i U |A| = \omega U M_i |A|, \quad \forall i \in \mathbb{X}, \quad (33)$$

*where  $\mathcal{M}(\rho) = \sum_{i \in \mathbb{X}} M_i \rho M_i^\dagger$  is an arbitrary Kraus decomposition of  $\mathcal{M}$ .*

*In particular, if  $\mathcal{M}$  is ergodic, then (up to a multiplicative constant)  $A$  must be of the form  $A = U\rho_*$ , where  $U$  is unitary and  $\rho_*$  is the unique fixed point state of  $\mathcal{M}$ .*

*Proof:* The eigenvalue condition  $\mathcal{M}(A) = \omega A$  can be reformulated as  $|A| = \bar{\omega} \sum_i (U^\dagger M_i U) |A| M_i^\dagger$ . Hence, we have

$$\begin{aligned} \text{Tr}[|A|] &= \bar{\omega} \sum_i \text{Tr}[(U^\dagger M_i U) |A| M_i^\dagger] \\ &\leq \sqrt{\sum_i \text{Tr}[(U^\dagger M_i U) |A| (U^\dagger M_i^\dagger U)] \cdot \sum_j \text{Tr}[M_j |A| M_j^\dagger]} \\ &= \sqrt{\text{Tr}[\mathcal{M}(U^\dagger |A| U)] \text{Tr}[\mathcal{M}(|A|)]} = \text{Tr}[|A|]. \end{aligned} \quad (34)$$

Since the Cauchy-Schwarz inequality is saturated, we necessarily have  $|A|^{1/2} M_i^\dagger = \omega |A|^{1/2} U^\dagger M_i^\dagger U$  for every  $i$ , or equivalently,  $M_i |A| = \bar{\omega} U^\dagger M_i U |A|$  for every  $i$  [in turn, this is equivalent to (33)]. Hence, we have

$$\begin{aligned} \mathcal{M}(|A|) &= \sum_i M_i |A| M_i^\dagger \\ &= \bar{\omega} \sum_i U^\dagger M_i U |A| M_i^\dagger \\ &= \bar{\omega} U^\dagger \mathcal{M}(A) = U^\dagger A = |A|, \end{aligned} \quad (35)$$

namely  $|A|$  is a fixed point of  $\mathcal{M}$ . The converse is just the statement of Lemma 5. Finally, when  $\mathcal{M}$  is ergodic, the fixed point  $|A|$  must be proportional to  $\rho_*$ , by Corollary 2. ■

**Remark 3.** *If  $\mathcal{M}$  is ergodic and  $\omega = 1$  then Corollary 2 implies that the unitary  $U$  in Theorem 5 must be the identity operator.*

**Remark 4.** *Note that Theorem 5 contains the statement that  $|A|$  is a fixed point of  $\mathcal{M}$  whenever  $A$  is an eigenvector of  $\mathcal{M}$  for some peripheral eigenvalue  $\omega$  with  $|\omega| = 1$ . This is the statement of Lemma 6 of [9], of which Theorem 5 provides an alternative derivation.*

Theorem 5 implies a rather counterintuitive fact: whenever we mix an ergodic channel with the identity channel we necessarily obtain a mixing channel!

**Corollary 5** (The mixture of an ergodic channel and the identity is mixing). *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be an ergodic channel. If the linear span  $\text{Span}_{\mathbb{C}}\{M_i\}_{i \in \mathbb{X}}$  contains the identity, then  $\mathcal{M}$  is mixing. In particular, if  $\mathcal{M}$  is an ergodic channel and  $\mathcal{I}$  is the identity channel, then*

$$\mathcal{M}' = p\mathcal{M} + (1-p)\mathcal{I} \quad (36)$$

is mixing for every  $p \in (0, 1)$ .

*Proof:* Let  $\omega$  be a peripheral eigenvalue of  $\mathcal{M}$ . Since  $\mathcal{M}$  is ergodic Theorem 5 implies  $M_i U \rho_* = \omega U M_i \rho_*$  for all  $i \in \mathcal{X}$ , with  $U \rho_*$  being the associated eigenoperator. By linearity this equation yields  $M U \rho_* = \omega U M \rho_*$ ,  $\forall M \in \text{Span}_{\mathbb{C}}\{M_i\}_{i \in \mathcal{X}}$ . Now, if the linear span contains the identity we have  $U \rho_* = \omega U \rho_*$ , and, therefore  $\omega = 1$ . Hence,  $\mu = 1$  is the only peripheral eigenvalue of  $\mathcal{M}$ . Since  $\mathcal{M}$  is ergodic, this implies that  $\mathcal{M}$  must be mixing. In particular, the channel (36) is ergodic by Theorem 4, and obviously contains the identity among its Kraus operators. ■

An alternative and instructive proof of the fact that convex combinations of the identity with an ergodic channel  $\mathcal{M}$  produce mixing maps can be derived as follows:

**Proposition 1** (Spectral properties of mixtures with the identity channel). *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a CPTP channel and  $p \in (0, 1)$  be a probability. The map  $\mathcal{M}' = p\mathcal{M} + (1-p)\mathcal{I}$  admits  $\omega = 1$  as its unique peripheral eigenvalue. Furthermore if  $\mathcal{M}$  is ergodic then  $\mathcal{M}'$  is mixing.*

*Proof:* If  $\omega = e^{i\beta}$  is a generic peripheral eigenvalue of  $\mathcal{M}'$  then by construction its associated eigenoperator  $A$  must satisfy the relation

$$\mathcal{M}(A) = \frac{\omega - (1-p)}{p} A, \quad (37)$$

where we have used the fact that  $p \neq 0$ . This in particular implies that  $A$  must also be an eigenoperator of  $\mathcal{M}$  belonging to an eigenvalue  $\lambda = [\omega - (1-p)]/p$ . Since  $\mathcal{M}$  is CPTP, however, we must have  $|\lambda| \leq 1$ , i.e.

$$\sqrt{1 + 2 \frac{1-p}{p^2} (1 - \cos \beta)} \leq 1, \quad (38)$$

which, since  $p \neq 1$ , can only be true if  $\beta = 0$ , i.e.  $\omega = 1$  and  $\lambda = 1$ . Moreover in case  $\mathcal{M}$  is ergodic we can invoke Corollary 2 to claim that  $A$  must be proportional to its fixed point state  $\rho_*$ . It then follows that (up to a proportionality constant) the only peripheral eigenvector of  $\mathcal{M}'$  coincides with  $\rho_*$ : the channel is hence mixing, with its stable point being  $\rho_*$ . ■

We conclude the section by observing that Theorem 5 allows one to give an alternative proof of the well-known fact about the stability of the property of mixing under convex randomizations:

**Theorem 6** (Stability of mixing under randomization). *If the channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  is mixing, then the channel  $\mathcal{M}_p := p\mathcal{M} + (1-p)\mathcal{M}'$  is mixing for every  $p \in (0, 1]$  and for every channel  $\mathcal{M}' \in \mathfrak{C}(\mathcal{H})$ .*

*Proof:* Suppose that  $\mathcal{M}(\rho) = \sum_{i \in \mathcal{X}} M_i \rho M_i^\dagger$  is mixing and let  $\rho_*$  be its fixed point. Then, for every operator  $U$ , the equation  $M_i U \rho_* = \omega U M_i \rho_*$ ,  $\forall i \in \mathcal{X}$  implies  $\omega = 1$  (otherwise  $\mathcal{M}$  would have a peripheral eigenvalue  $\omega \neq 1$ ). Moreover, since  $\mathcal{M}$  is mixing, we know from Theorem 4 that  $\mathcal{M}_p$  is ergodic for every  $p \in (0, 1]$ . Hence, from Theorem 5 it follows that any peripheral eigenvector of  $\mathcal{M}_p$  is of the form

$A = U \rho_{p,*}$ , where  $\rho_{p,*}$  is the fixed point of  $\mathcal{M}_p$  and  $U$  is a unitary satisfying  $M_{p,i} U \rho_{p,*} = \omega U M_{p,i} \rho_{p,*}$ ,  $\forall i \in \mathcal{X}_p$ , with  $\{M_{p,i}\}_{i \in \mathcal{X}_p}$  being the Kraus operators of  $\mathcal{M}_p$ . Multiplying by the inverse of  $\rho_{p,*}$  on its support, we then obtain

$$M_{p,i} U Q_p = \omega U M_{p,i} Q_p, \quad \forall i \in \mathcal{X}_p, \quad (39)$$

where  $Q_p$  is the projector on the support of  $\rho_{p,*}$ . Now, since by Theorem 4 the support of  $\rho_*$  is contained in the support of  $\rho_{p,*}$ , we have  $Q_p \rho_* = \rho_*$ , and therefore,

$$M_{p,i} U \rho_* = \omega U M_{p,i} \rho_*, \quad \forall i \in \mathcal{X}_p. \quad (40)$$

Since there is a Kraus form for  $\mathcal{M}_p$  that includes all the Kraus operators of  $\mathcal{M}$ , this implies in particular  $M_i U \rho_* = \omega U M_i \rho_*$ ,  $\forall i \in \mathcal{X}$ . From the fact that  $\mathcal{M}$  is mixing we conclude that  $\omega = 1$ . Hence,  $\mathcal{M}_p$  is mixing. ■

## VI. ERGODICITY AND MIXING FOR CHANNELS WITH FAITHFUL FIXED POINT

The general characterization of ergodic channels, provided by Theorem 1, becomes more specific in the case of channels with a *faithful* fixed point, namely a fixed point state  $\rho_*$  with  $\text{Supp}(\rho_*) = \mathcal{H}$ , or, equivalently  $\rho_* > 0$ . Ergodic channels with faithful fixed point are also known as *irreducible quantum channels* [8], [30]. In the same references, mixing channels with faithful fixed point state are referred to as *primitive*.

**Theorem 7** (Ergodicity and proper invariant subspaces). *If a channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  has a faithful fixed point state  $\rho_* > 0$ , then the followings are equivalent:*

- 1)  $\mathcal{M}$  is ergodic;
- 2)  $\mathcal{M}$  has no proper invariant subspace;
- 3)  $\mathcal{M}^\dagger$  has no proper invariant subspace.

*Proof:* 1)  $\iff$  2) Suppose that  $\mathcal{M}$  has a proper invariant subspace  $S \subset \mathcal{H}$ . Then, by Lemma 2,  $\mathcal{M}$  must have a fixed point  $\rho_S$  with  $\text{Supp}(\rho_S) \subseteq S$ . Hence,  $\mathcal{M}$  has two distinct fixed points  $\rho_S$  and  $\rho_*$ , that is,  $\mathcal{M}$  is not ergodic. Conversely, if  $\mathcal{M}$  is not ergodic, then it has two orthogonal invariant subspaces (Theorem 1), which, by definition, are proper subspaces of  $\mathcal{H}$ . [It is worth stressing that the implication 2)  $\Rightarrow$  1) does not require  $\rho_*$  to be of full rank]. 2)  $\iff$  3) Immediate from Lemma 4. ■

Not having a proper invariant subspace is an important algebraic property, which is equivalent to the irreducibility of the matrix algebra generated by the Kraus operators:

**Theorem 8** (Ergodicity and irreducibility of matrix algebras). *For a completely positive (not necessarily trace-preserving) map  $\mathcal{M}$ , the followings are equivalent:*

- 1)  $\mathcal{M}$  has no proper invariant subspace  $S \subset \mathcal{H}$ ;
- 2) the matrix algebra  $\mathcal{A}_{\mathcal{M}}$  generated by the Kraus operators  $\{M_i\}_{i \in \mathcal{X}}$  is irreducible;
- 3) for every vector  $\varphi \in \mathcal{H}$ , the set of vectors  $\{M_{i_1} M_{i_2} \cdots M_{i_N} |\varphi\rangle \mid (i_1, i_2, \dots, i_N) \in \mathcal{X}^{\times N}, N \in \mathbb{N}\}$  spans the whole Hilbert space.

*Proof:* 1)  $\iff$  2) By definition,  $\mathcal{A}_{\mathcal{M}}$  is reducible if and only if it has a proper invariant subspace



$S \subset \mathcal{H}$ . 1)  $\iff$  3) The span of the vectors  $\{M_{i_1}M_{i_2}\cdots M_{i_N}|\varphi\rangle \mid (i_1, i_2, \dots, i_N) \in X^{\times N}, N \in \mathbb{N}\}$  is an invariant subspace  $S$ . If  $\mathcal{M}$  has no proper invariant subspace, then  $S = \mathcal{H}$ . Conversely, if  $\mathcal{M}$  has a proper invariant subspace  $S \subset \mathcal{H}$ , then given  $|\varphi\rangle \in S$  the vectors  $\{M_{i_1}M_{i_2}\cdots M_{i_N}|\varphi\rangle \mid (i_1, i_2, \dots, i_N) \in X^{\times N}\}$  can only span a subset of  $S$ . ■

**Remark 5.** Note that Theorem 8 applies both to  $\mathcal{M}$  and to  $\mathcal{M}^\dagger$ : the two algebras  $\mathcal{A}_\mathcal{M}$  and  $\mathcal{A}_{\mathcal{M}^\dagger}$  must be both irreducible.

**Remark 6.** Condition 3) in Theorem 8 states that an arbitrary input state  $|\varphi\rangle \in \mathcal{H}$  evolving under the discrete-time dynamics  $\mathcal{M}$  will generate a stochastic trajectory of pure states that span the whole Hilbert space. Such a property is the quantum analogue of the classical property that the trajectory of an ergodic dynamical system fills up the whole state space, up to a set of zero measure.

**Remark 7.** In fact, condition 3) in Theorem 8 can be refined by showing that only a finite number  $N$  of iterations of the channel are enough for the trajectory of a arbitrary pure state to span the whole Hilbert space. The number  $N$  is upper bounded by the quantum Wielandt's inequality [30], which in our notation reads  $N \leq d^2(d^2 - |X| - 1)$ , where  $|X|$  is the number of Kraus operators.

Further information about the structure of an ergodic map with faithful fixed point  $\mathcal{M}$  can be extracted from the analysis of its peripheral eigenvalues and eigenvectors. This analysis is the subject of a quantum generalization of the Perron-Frobenius theory of classical Markov chains which was initiated in [21] and further developed for the case of positive maps in [8]. The following theorem summarizes some of these results and takes advantage of the complete positivity of quantum channels to give a convenient condition for the peripheral eigenvectors in terms of the Kraus operators:

**Theorem 9** (Peripheral eigenvectors of quantum channels with faithful fixed point). *If  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  is an ergodic channel with faithful fixed point state  $\rho_* > 0$ , then*

- 1) *an operator  $A_\omega \in \mathcal{B}(\mathcal{H})$  is an eigenvector of  $\mathcal{M}$  with eigenvalue  $\omega$  is and only if (up to a proportionality constant)  $A_\omega = U_\omega \rho_*$ , where  $U_\omega$  is a unitary operator satisfying*

$$M_i U_\omega = \omega U_\omega M_i, \quad \forall i \in X, \quad (41)$$

*$\mathcal{M}(\rho) = \sum_{i \in X} M_i \rho M_i^\dagger$  being an arbitrary Kraus decomposition of  $\mathcal{M}$ ;*

- 2) *the peripheral eigenvalues of  $\mathcal{M}$  are roots of the unit and form a finite cyclic group  $F = \{e^{2\pi i l/L} \mid l = 0, \dots, L-1\}$  with  $L \leq d^2$ ;*
- 3) *every peripheral eigenvalue is non-degenerate;*
- 4) *the unitaries  $\{U_\omega \mid \omega \in F\}$  form a unitary representation of the cyclic group  $F$ ;*
- 5) *if  $\omega$  is a peripheral eigenvalue of  $\mathcal{M}^\dagger$ , then the corresponding eigenvector is (proportional to) a unitary;*

- 6)  *$\omega$  is a peripheral eigenvalue of  $\mathcal{M}$  with eigenvector  $U_\omega \rho_*$  if and only if  $\bar{\omega}$  is a peripheral eigenvalue of  $\mathcal{M}^\dagger$  with eigenvector  $U_\omega$ .*

*Proof:* 1) The key of the entire derivation is the identity (41): we hence start proving it. From Theorem 5, we know that  $A_\omega$  is an eigenvector of  $\mathcal{M}$  with eigenvalue  $\omega$  if and only if  $A_\omega = U_\omega \rho_*$  for some unitary  $U_\omega$  such that  $M_i U_\omega \rho_* = \omega U_\omega M_i \rho_*$  for every  $i \in X$ . Since  $\rho_*$  is invertible, this condition is equivalent to (41).

2) Equation (41) allows us to show that if  $\omega$  is an eigenvalue with eigenvector  $A_\omega = U_\omega \rho_*$ , then also its inverse  $\bar{\omega}$  is an eigenvalue of  $\mathcal{M}$ , with eigenvector  $A_{\bar{\omega}} := U_\omega^\dagger \rho_*$ . In addition, if  $\omega_1$  and  $\omega_2$  are two eigenvalues, with unitaries  $U_{\omega_1}$  and  $U_{\omega_2}$ , respectively, then also  $\omega_1 \omega_2$  is an eigenvalue, with unitary  $U_{\omega_1} U_{\omega_2}$  (multiplicative rules). This proves that the eigenvalues of  $\mathcal{M}$  must form a group  $F$ . Clearly, the group has order  $|F| \leq d^2$ , because the eigenvectors belong to the  $d^2$ -dimensional vector space  $\mathcal{B}(\mathcal{H})$ . Moreover, since the peripheral eigenvalues lie on the unit circle,  $F$  must be a cyclic group, consisting of powers of some generator  $\omega_1 = e^{2\pi i/L}$  for some integer number  $L \leq d^2$ .

3) The multiplicative rules in the previous point imply that the operator  $A_\omega = U_\omega \rho_*$  with  $U_\omega$  defined as in (41) is the only eigenvector of  $\mathcal{M}$  with eigenvalue  $\omega$ . Indeed if there were two of them, we would have two unitaries  $U_\omega$  and  $V_\omega$  satisfying (41). Therefore by the multiplicative properties discussed above also the operator  $B = V_\omega^\dagger U_\omega \rho_*$  would be a fixed point for  $\mathcal{M}$ . Since  $\mathcal{M}$  is ergodic, we must have  $B = \lambda \rho_*$  for some proportionality constant  $\lambda \in \mathbb{C}$ . Hence,  $V_\omega = \lambda U_\omega$ .

4) Let  $A_{\omega_1} = U_{\omega_1} \rho_*$  be the eigenvector corresponding to the eigenvalue  $\omega_1 := e^{2\pi i/L}$ . Then, by the multiplicative rules of point 1) it follows that  $A_{\omega_l} := U_{\omega_1}^l \rho_*$  is an eigenvector corresponding to the eigenvalue  $\omega_l := e^{2\pi i l/L}$ , for every  $l \in \mathbb{N}$ . Since the eigenvalues are non-degenerate,  $U_{\omega_1}^l \rho_*$  is actually the eigenvector corresponding to the eigenvalue  $e^{2\pi i l/L}$  (up to a multiplicative constant). In particular, since  $\omega_L = 1$ , we must have  $U_{\omega_1}^L = e^{i\alpha} I$ , for some phase  $\alpha \in [0, 2\pi)$ . Now, without loss of generality the phase  $\alpha$  can be chosen to be 0: indeed, we can always re-defining  $U_{\omega_1}$  to be  $U'_{\omega_1} := e^{i\alpha/L} U_{\omega_1}$ . With this choice, the correspondence  $\omega_l \mapsto U_{\omega_l}$  is a unitary representation of the group  $F$ .

5) The thesis follows by noticing that if  $\bar{\omega}$  is a peripheral eigenvalue of  $\mathcal{M}^\dagger$  with eigenvector  $B$  then Lemma 3 implies that  $\omega$  must be an eigenvalue of  $\mathcal{M}$  with eigenvector  $B \rho_*$ . Since  $\omega$  is peripheral by construction and  $\rho_*$  is faithful, we can prove the thesis by invoking point 1) to say that there must exist  $U_\omega$  unitary such that (up to a proportionality constant)  $B \rho_* = U_\omega \rho_*$ . But this immediately implies that  $B$  must be proportional to  $U_\omega$ .

6) The thesis follows by noticing that if  $\omega$  is a peripheral eigenvalue of  $\mathcal{M}$  then point 1) says that its eigenvector can be written as  $A_\omega = U_\omega \rho_*$  (up to a proportionality constant) with  $U_\omega$  satisfying (41). From this it immediately follows that  $\bar{\omega}$  is an eigenvalue of  $\mathcal{M}^\dagger$  with eigenvector  $U_\omega$ : indeed, we have  $\mathcal{M}^\dagger(U_\omega) = \sum_i M_i^\dagger (U_\omega M_i) = \sum_i M_i^\dagger (\bar{\omega} M_i U_\omega) = \bar{\omega} \mathcal{M}^\dagger(I) U_\omega = \bar{\omega} U_\omega$ . Note that the converse is guaranteed by Lemma 3: if  $U_\omega$  is an eigenvector of  $\mathcal{M}^\dagger$  with eigenvalue  $\bar{\omega}$ ,

then  $U_\omega \rho_*$  is an eigenvector of  $\mathcal{M}$  with eigenvalue  $\omega$ . ■

The fact that the peripheral eigenvalues of an ergodic channel with faithful fixed point are  $L$ th roots of the unit for some  $L \in \{1, \dots, d^2\}$  was known from [8] and [21]. However, the condition in terms of Kraus operators in Theorem 9 allows us to prove a slightly stronger result, namely that the peripheral eigenvalues of an ergodic random-unitary channel are  $d$ -th roots of the unit:

**Corollary 6** (Peripheral eigenvalues of random-unitary channels are  $d$ th roots of the unit). *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be an ergodic channel with faithful fixed point state and let  $\omega \in \mathbb{C}$  be a peripheral eigenvalue of  $\mathcal{M}^\dagger$  with  $|\omega| = 1$ . If  $\text{Span}_{\mathbb{C}}\{M_i\}_{i \in \mathcal{X}}$  contains an invertible operator, then  $\omega^d = 1$ . In particular, if  $\mathcal{M}$  is a random-unitary channel, of the form  $\mathcal{M}(\rho) = \sum_i p_i U_i \rho U_i^\dagger$ , we have  $\omega^d = 1$ .*

*Proof:* Let  $A_\omega = U_\omega \rho_*$  ( $U_\omega$  unitary) be the eigenvector for the eigenvalue  $\omega$ . Suppose that  $\text{Span}_{\mathbb{C}}\{M_i\}_{i \in \mathcal{X}}$  contains an invertible operator  $M$ . By linearity, the eigenvalue condition (41) gives  $MU = \omega UM$ . Taking the determinant on both sides we obtain  $\det(M) \det(U) = \omega^d \det(M) \det(U)$ . Since  $M$  and  $U$  are invertible, we have  $\det(M) \neq 0$  and  $\det(U) \neq 0$ . Hence,  $\omega^d = 1$ . ■

Using the fact that the peripheral eigenvalues are roots of unit we obtain a characterization of mixing channels with a faithful fixed point. The interesting feature of this characterization is that it connects the two properties of mixing and ergodicity:

**Theorem 10** (Connection between mixing and ergodicity). *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be an ergodic channel with faithful fixed point state. The channel  $\mathcal{M}$  is mixing if and only if the channel  $\mathcal{M}^k$  is ergodic for all  $k \leq d^2$ . Moreover, if  $\text{Span}_{\mathbb{C}}\{M_i\}_{i \in \mathcal{X}}$  contains an invertible operator, then  $\mathcal{M}$  is mixing if and only if  $\mathcal{M}^d$  is ergodic. In particular, a random-unitary channel  $\mathcal{M}$  is mixing if and only if  $\mathcal{M}^d$  is ergodic.*

*Proof:* If  $\mathcal{M}$  is mixing, then also  $\mathcal{M}^k$  must be mixing for every  $k$ , and, therefore, ergodic. Conversely, suppose that  $\mathcal{M}^k$  is ergodic for all  $k \leq d^2$  and assume by contradiction that  $\mathcal{M}$  is not mixing, namely that  $\mathcal{M}$  has a peripheral eigenvalue  $\omega \neq 1$  for some eigenvector  $A_\omega$  which is not a multiple of  $\rho_*$ . By Theorem 9 it follows that there exists an  $L \leq d^2$  such that  $\omega^L = 1$ . Since  $\mathcal{M}^L(A_\omega) = \omega^L A_\omega = A_\omega$ , this means that  $\mathcal{M}^L$  has two distinct fixed points  $A_\omega$  and  $\rho_*$ , namely, it is not ergodic, in contradiction with the hypothesis. ■

## VII. ERGODICITY AND MIXING FOR UNITAL CHANNELS

Here we focus our attention on a particular type of channels, unital channels, which includes a large number of physically interesting examples, and allows for an even more specific characterization of ergodicity and mixing.

We remind that a channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  is called *unital* if and only if it preserves the identity, that is, if and only if  $\mathcal{M}(I) = I$ . The easiest example of unital channels is given by the class of *random-unitary* channels, of the form  $\mathcal{M}(\rho) = \sum_i p_i U_i \rho U_i^\dagger$ , with  $U_i$  unitary operator and  $p_i \geq 0$  for every  $i$ .

Note, however, that there are many examples of unital channels that are not of the random-unitary form [29], [31], [32]: as a matter of fact, as anticipated in Section III-B the two sets coincides only for qubit systems.

A first useful observation is that, in the case of peripheral eigenvalues, the eigenvectors of a unital channel are also eigenvectors of its adjoint:

**Lemma 6.** *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a unital channel and  $\omega \in \mathbb{C}$  be a complex number on the unit circle  $|\omega| = 1$ . Then  $A$  is an eigenvector of  $\mathcal{M}$  belonging to eigenvalue  $\omega$  if and only if  $A$  is also eigenvector of the adjoint map  $\mathcal{M}^\dagger$  belonging to eigenvalue  $\bar{\omega}$ .*

*Proof:* Immediate consequence of Lemma 3 and of the fact that  $\mathcal{M}(I) = \mathcal{M}^\dagger(I) = I$ . ■

Note that unitality is an essential ingredient for the Corollary: if  $\mathcal{M}$  is not unital the fixed points of  $\mathcal{M}$  can easily differ from the fixed points of  $\mathcal{M}^\dagger$ . For example consider the case of the erasure channel of (18) with  $\rho_0 \neq I/d$ . While  $\rho_0$  is clearly a fixed point state for  $\mathcal{M}$ , it is not an eigenvector for its adjoint map  $\mathcal{M}^\dagger(A) = I \text{Tr}[\rho_0 A]$ .

A full characterization of the peripheral eigenvalues of unital channels is the following:

**Theorem 11** (Peripheral eigenvalues of unital channels). *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a unital channel and  $\omega \in \mathbb{C}$  be a peripheral eigenvalue of  $\mathcal{M}$  with  $|\omega| = 1$ . An operator  $A \in \mathcal{B}(\mathcal{H})$  is an eigenvector of  $\mathcal{M}$  belonging to eigenvalue  $\omega$  if and only if, for every Kraus representation  $\mathcal{M}(\rho) = \sum_{i \in \mathcal{X}} M_i \rho M_i^\dagger$  we have*

$$M_i A = \omega A M_i, \quad \forall i \in \mathcal{X}. \quad (42)$$

*Moreover, the eigenspace of  $\mathcal{B}(\mathcal{H})$  corresponding to  $\omega$  is spanned by partial isometries.*

*Proof:* The proof is provided in the Appendix. ■

### A. Algebraic Characterization of Ergodic Unital Channels

**Theorem 12** (Characterization of ergodicity for unital channels). *For a unital channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  the followings are equivalent:*

- 1)  $\mathcal{M}$  is ergodic;
- 2) there exists no projector  $0 < P < I$  such that  $\mathcal{M}(P) = P$  [or equivalently,  $\mathcal{M}^\dagger(P) = P$ ];
- 3) the matrix algebra  $\mathcal{A}_{\mathcal{M}, \mathcal{M}^\dagger}$  generated by the Kraus operators  $\{M_i\}_{i \in \mathcal{X}}$  and  $\{M_i^\dagger\}_{i \in \mathcal{X}}$  is irreducible;
- 4)  $\mathcal{A}_{\mathcal{M}, \mathcal{M}^\dagger} = \mathcal{B}(\mathcal{H})$ ;
- 5) if an operator  $A \in \mathcal{B}(\mathcal{H})$  commutes with all operators in  $\mathcal{A}_{\mathcal{M}, \mathcal{M}^\dagger}$  (or equivalently, with all Kraus operators and with their adjoints), then  $A$  is multiple of the identity.

*Proof:* 1)  $\Rightarrow$  2) If  $\mathcal{M}(P) = P$  for some  $P < I$ , then  $\mathcal{M}$  cannot be ergodic because it would have two distinct fixed points  $P$  and  $I$ . 2)  $\Rightarrow$  3) If  $\mathcal{A}_{\mathcal{M}, \mathcal{M}^\dagger}$  is reducible, then it has a proper invariant subspace  $S \subset \mathcal{H}$ , and denoting by  $P < I$  the projector on  $S$  we have  $M_i P = P M_i P$  and  $M_i^\dagger P = P M_i^\dagger P$  for every  $i \in \mathcal{X}$ . From these relations we get  $M_i P = P M_i$ ,  $\forall i$  and therefore  $\mathcal{M}(P) = \sum_i M_i P M_i^\dagger = \sum_i M_i M_i^\dagger P =$

$\mathcal{M}(I)P = P$ . 3)  $\iff$  4)  $\iff$  5) The equivalence of 3), 4), and 5) is a standard fact for operator algebras that are closed under adjoint. 5)  $\Rightarrow$  1) Let  $A$  be a fixed point of  $\mathcal{M}$ . By Theorem 11, we have  $M_i A = A M_i$ ,  $\forall i \in \mathbb{X}$ . Since  $A$  is also a fixed point of  $\mathcal{M}^\dagger$  we also have  $M_i^\dagger A = A M_i^\dagger$ ,  $\forall i \in \mathbb{X}$ . In conclusion,  $A$  commutes with the Kraus operators  $\{M_i\}_{i \in \mathbb{X}}$  and  $\{M_i^\dagger\}_{i \in \mathbb{X}}$ , and hence, with all the algebra  $\mathcal{A}_{\mathcal{M}, \mathcal{M}^\dagger}$  generated by them. Hence,  $A$  is a multiple of the identity. Since every fixed point of  $\mathcal{M}$  is proportional to the identity, we conclude that  $\mathcal{M}$  is ergodic.  $\blacksquare$

Specializing to the case of random-unitary channels, Theorem 12 gives the interesting group-theoretic characterization:

**Corollary 7** (Group-theoretic characterization of ergodicity for random-unitary channels). *A random-unitary channel  $\mathcal{M} = \sum_{i \in \mathbb{X}} p_i \mathcal{U}_i$  with  $\mathcal{U}_i(\rho) = U_i \rho U_i^\dagger$  is ergodic if and only if the group representation generated by the unitaries  $\{U_i\}_{i \in \mathbb{X}}$  is irreducible.*

For example, the qubit channel  $\mathcal{M} \in \mathfrak{C}(\mathbb{C}^2)$  defined by  $\mathcal{M}(\rho) := pX\rho X + (1-p)Y\rho Y$ , where  $X$  and  $Y$  are Pauli matrices, is ergodic for every value  $p \in (0, 1)$ . Indeed,  $X$  and  $Y$  are sufficient to generate the Pauli group  $G = \{\pm I, \pm X, \pm Y, \pm Z\}$ , which acts irreducibly on  $\mathbb{C}^2$ . The same consideration applies in dimension  $d > 2$  for the channel  $\mathcal{M} \in \mathfrak{C}(\mathbb{C}^d)$  defined by  $\mathcal{M}(\rho) := pS\rho S^\dagger + (1-p)M\rho M^\dagger$ , where  $S$  and  $M$  are the shift and multiply operators, defined relative to an orthonormal basis  $\{|n\rangle\}_{n=0}^{d-1}$  as  $S|n\rangle = |(n+1) \bmod d\rangle$  and  $M|n\rangle = \omega^n|n\rangle$ ,  $\omega = e^{2\pi i/d}$ , respectively.

### B. From Ergodicity to Mixing in the Case of Unital Channels

We now give a sufficient condition for mixing, which has a nice algebraic form and connects mixing with ergodicity. Unfortunately, in general this is only a sufficient condition. However, the condition is also necessary for a particular class of channels, here called *diagonalizable channels*.

**Definition 6.** A channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  is called *diagonalizable* if  $\mathcal{M}\mathcal{M}^\dagger = \mathcal{M}^\dagger\mathcal{M}$ .

The reason for the name is that a channel  $\mathcal{M}$  satisfies the relation  $\mathcal{M}\mathcal{M}^\dagger = \mathcal{M}^\dagger\mathcal{M}$  if and only if it is diagonalizable as a linear operator on  $\mathcal{B}(\mathcal{H})$ , that is, if and only if there exists a set of complex eigenvalues  $\{\mu_i \in \mathbb{C}\}$  an orthonormal basis for  $\mathcal{B}(\mathcal{H})$  consisting of operators  $\{\Phi_i\}_{i=1}^{d^2}$  such that  $\langle \Phi_i, \Phi_j \rangle = \delta_{ij}$  and  $\mathcal{M}(\rho) = \sum_i \mu_i \Phi_i \langle \Phi_i, \rho \rangle$ . For example, all Pauli channels are diagonalizable. More generally, all generalized Pauli channels in dimension  $d$ , consisting of random mixtures of unitaries in the discrete Weyl-Heisenberg representation, are diagonalizable.

To give our condition for mixing we need to introduce the notion of *square modulus* of a unital channel:

**Definition 7.** The square modulus of a unital channel  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  is the unital channel  $\mathcal{M}^\dagger\mathcal{M} \in \mathfrak{C}(\mathcal{H})$ .

By definition, the square modulus is a self-adjoint non-negative operator:  $\langle A, \mathcal{M}^\dagger\mathcal{M}(A) \rangle = \langle \mathcal{M}(A), \mathcal{M}(A) \rangle \geq 0$  for

every  $A \in \mathcal{B}(\mathcal{H})$ . This implies that  $\mathcal{M}^\dagger\mathcal{M}$  can be diagonalized and has only non-negative eigenvalues. Hence, it is clear that the properties of ergodicity and mixing coincide for square modulus: a square modulus is ergodic if and only if it is mixing. Now, the main theorem of this section is the following:

**Theorem 13** (Mixing from the ergodicity of the square modulus). *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a unital channel. A sufficient condition for  $\mathcal{M}$  to be mixing is that the square modulus  $\mathcal{M}^\dagger\mathcal{M}$  is ergodic. If  $\mathcal{M}$  is diagonalizable, then the condition is also necessary.*

*Proof:* Suppose that the channel  $\mathcal{M}^\dagger\mathcal{M}$  is ergodic. Then, the channel  $\mathcal{M}$  must be mixing. Indeed, if an operator  $A \in \mathcal{B}(\mathcal{H})$  is an eigenvector of  $\mathcal{M}$  with eigenvalue  $\omega$  on the unit circle, say  $\mathcal{M}(A) = \omega A$ , then by Corollary 6 we have  $\mathcal{M}^\dagger(A) = \bar{\omega} A$  and  $\mathcal{M}^\dagger\mathcal{M}(A) = A$ . Since  $\mathcal{M}^\dagger\mathcal{M}$  is ergodic, this implies  $A \propto I$ , and therefore,  $\omega = 1$ . In conclusion, we proved that the only peripheral eigenvalue of  $\mathcal{M}$  is  $\mu = 1$  and is non-degenerate, that is,  $\mathcal{M}$  is mixing. Conversely, suppose that  $\mathcal{M}$  is diagonalizable and mixing. Then, also  $\mathcal{M}^\dagger\mathcal{M}$  must be mixing: indeed, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} (\mathcal{M}^\dagger\mathcal{M})^n(\rho) &= \lim_{n \rightarrow \infty} \mathcal{M}^{\dagger n} \mathcal{M}^n(\rho) \\ &= \lim_{m \rightarrow \infty} \mathcal{M}^{\dagger m} \left( \lim_{n \rightarrow \infty} \mathcal{M}^n(\rho) \right) \\ &= \lim_{m \rightarrow \infty} \mathcal{M}^{\dagger m}(I/d) = I/d. \end{aligned} \quad (43)$$

Since  $\mathcal{M}^\dagger\mathcal{M}$  must be mixing, it must also be ergodic.  $\blacksquare$

Note that the ergodicity of the square modulus is a necessary condition for mixing only in the case of diagonalizable channels. For example, consider the (non-diagonalizable) channel

$$\mathcal{M}(\rho) = \sum_{n=1}^d |e_n\rangle \langle n|\rho|n\rangle \langle e_n|, \quad (44)$$

where  $\{|e_n\rangle\}_{n=1}^d$  is the Fourier basis ( $|e_n\rangle := d^{-1/2} \sum_{k=1}^d e^{2\pi i k n/d} |n\rangle$ ). It is easy to see that  $\mathcal{M}$  is mixing, and, in fact, that  $\mathcal{M}^2(\rho) = I/d$  for every state  $\rho$ . However,  $\mathcal{M}^\dagger\mathcal{M}$  is not ergodic: we have  $\mathcal{M}^\dagger\mathcal{M}(\rho) = \sum_{n=1}^d |n\rangle \langle n|\rho|n\rangle \langle n|$  and every projector  $P_n = |n\rangle \langle n|$  is a fixed point of  $\mathcal{M}^\dagger\mathcal{M}$ .

Specializing to random-unitary channels, Theorem 13 becomes:

**Corollary 8** (Ergodicity of the square modulus for random-unitary channels). *A random-unitary channel  $\mathcal{M} = \sum_{i \in \mathbb{X}} p_i \mathcal{U}_i$ , with  $\mathcal{U}_i(\rho) = U_i \rho U_i^\dagger$ , is mixing if the group representation generated by the unitaries  $\{U_i^\dagger U_j\}_{i,j \in \mathbb{X}}$  is irreducible. In particular, a random-unitary qubit channel  $\mathcal{M}$  is mixing if the unitaries  $\{U_i^\dagger U_j\}_{i,j \in \mathbb{X}}$  do not commute. These conditions are also necessary in the case of diagonalizable random-unitary channels.*

For example, while the diagonalizable qubit channel  $\mathcal{M}(\rho) = pX\rho X + (1-p)Y\rho Y$  is ergodic for every  $p \in (0, 1)$ , it is clearly not mixing because in this case the set  $\{U_i^\dagger U_j\}_{i,j \in \mathbb{X}}$  consists of the commuting unitaries  $\{I, \pm iZ\}$ .



On the other hand, the diagonalizable qubit channel  $\mathcal{N}(\rho) = p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z$  is mixing for every choice of probabilities  $(p_x, p_y, p_z) \in (0, 1)^{\times 3}$ .

Incidentally, we note that the ergodicity of  $\mathcal{M}^\dagger \mathcal{M}$  is equivalent to a condition discussed by Streater [1]:

**Proposition 2.** *Let  $\mathcal{M} \in \mathfrak{C}(\mathcal{H})$  be a unital channel. The square modulus of  $\mathcal{M}$  is ergodic if and only if there is no projector  $P < I$  and no unitary  $U$  such that  $\mathcal{M}(P) = U P U^\dagger$ .*

*Proof:* The proof can be found in the Appendix. ■

The above Proposition shows that Streater's condition is sufficient for mixing, and also necessary in the case of diagonalizable channels.

### VIII. ERGODIC SEMIGROUPS AND ERGODIC CHANNELS

In this final section we analyze the ergodicity property of dynamical semigroup showing how this problem can be related to the ergodicity problem of CPTP transformations.

#### A. Ergodic Channels as Generators for Mixing CPTP Semigroups

It has recently been pointed out [28] that any CPTP channel  $\mathcal{M}$  can be used to induce a continuous CPTP semigroup evolution on  $\mathfrak{S}(\mathcal{H})$ , described by a Markovian master equation

$$\dot{\rho}(t) = \mathcal{L}^{(\mathcal{M})}(\rho(t)), \quad \forall t \geq 0. \quad (45)$$

For instance this can be done by identifying the generator  $\mathcal{L}^{(\mathcal{M})}$  of the above equation with the superoperator

$$\mathcal{L}^{(\mathcal{M})} = \gamma(\mathcal{M} - \mathcal{I}), \quad (46)$$

where  $\gamma > 0$  is a constant, which scales the unit of time.<sup>1</sup> It is important to stress that the continuous trajectories  $\rho(t)$  defined by (45) in general have nothing to do with the discrete trajectories introduced in (10). Indeed for the continuous case one has  $\rho(t) = \mathcal{T}_t^{(\mathcal{M})}(\rho(0))$ , where, for  $t \geq 0$ ,  $\mathcal{T}_t^{(\mathcal{M})}$  is an element of the semigroup of CPTP maps defined by

$$\mathcal{T}_t^{(\mathcal{M})} = e^{\mathcal{L}^{(\mathcal{M})}t} = e^{-\gamma t} e^{\gamma t \mathcal{M}}, \quad (47)$$

whose properties may be rather different from those of the maps  $\mathcal{M}^n$ . One can easily verify that if  $\mathcal{M}$  is ergodic, then  $\mathcal{L}^{(\mathcal{M})}$  admits a unique eigenvector associated with the null eigenvalue (up to a multiplicative factor), i.e.

$$\mathcal{L}^{(\mathcal{M})}(A) = 0 \iff A = c\rho_*, \quad (48)$$

where  $c$  is a complex number and  $\rho_*$  is the fixed point state of  $\mathcal{M}$ . Indeed from (46) it follows that the eigenvalues of  $\mathcal{L}$  must be of the form  $\mu = \gamma(\lambda - 1)$  with  $\lambda$  being the eigenvalues of  $\mathcal{M}$ . The condition  $\mu = 0$  hence implies  $\lambda = 1$ , which according to Lemma 1 is only possible if the eigenvector is of the form described in (48). Accordingly [4], [33], [34] in the

<sup>1</sup>To see this, simply observe that  $\mathcal{L}$  can be cast in an explicit Lindblad form [2] by introducing a set of Kraus operators  $\{M_i\}_{i \in X}$  for  $\mathcal{M}$  and writing  $\mathcal{I}(A) = A = \frac{1}{2} \left( \sum_{i \in X} M_i^\dagger M_i \right) A + \frac{1}{2} A \left( \sum_{i \in X} M_i^\dagger M_i \right)$ .

limit of  $t \rightarrow \infty$ , the channel  $\mathcal{T}_t^{(\mathcal{M})}$  brings all the input states toward the fixed point state of  $\mathcal{M}$ , i.e.

$$\lim_{t \rightarrow \infty} \|\mathcal{T}_t^{(\mathcal{M})}(\rho) - \rho_*\|_1 = 0, \quad \forall \rho \in \mathfrak{S}(\mathcal{H}), \quad (49)$$

implying that (for  $t > 0$ ) each of the maps  $\mathcal{T}_t^{(\mathcal{M})}$  is *mixing*. As an example consider the case of the ergodic (but not mixing) qubit channel defined in (14). In this case for  $n \geq 1$  integer we have

$$\mathcal{M}^{2n+1} = \mathcal{M}, \quad \mathcal{M}^{2n} = \mathcal{M}_D, \quad (50)$$

with  $\mathcal{M}_D(\rho) := \langle 0|\rho|0\rangle|0\rangle\langle 0| + \langle 1|\rho|1\rangle|1\rangle\langle 1|$  being the fully depolarizing channel. Therefore,

$$\mathcal{T}_t^{(\mathcal{M})}(\rho) = \frac{(1 - e^{-\gamma t})^2}{2} \mathcal{M}_D(\rho) + \frac{1 - e^{-2\gamma t}}{2} \mathcal{M}(\rho) + e^{-\gamma t} \rho \quad (51)$$

which in the limit of large  $t$  converges to the fixed point  $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$  of  $\mathcal{M}$ , independently of the input  $\rho$ .

Suppose hence we have another semigroup generator of the type defined in (46), i.e.

$$\mathcal{L}^{(\mathcal{E})} = \kappa(\mathcal{E} - \mathcal{I}), \quad (52)$$

with  $\mathcal{E}$  being a (not necessarily ergodic) element of  $\mathfrak{C}(\mathcal{H})$  and  $\kappa$  being a nonnegative constant, and consider the evolution induced on  $S$  by the contemporary action of  $\mathcal{L}^{(\mathcal{M})}$  and  $\mathcal{L}^{(\mathcal{E})}$ , i.e.

$$\dot{\rho}(t) = (\mathcal{L}^{(\mathcal{M})} + \mathcal{L}^{(\mathcal{E})})(\rho(t)), \quad \forall t \geq 0. \quad (53)$$

Introducing the parameters  $\tilde{\gamma} = \gamma + \kappa$  and  $\lambda = \gamma/\tilde{\gamma} \in (0, 1]$  we observe that the resulting Lindblad superoperator can be cast again in the form (46), for the CPTP map  $\tilde{\mathcal{M}} = \lambda\mathcal{M} + (1-\lambda)\mathcal{E}$ , i.e.

$$\mathcal{L}^{(\mathcal{M})} + \mathcal{L}^{(\mathcal{E})} = \mathcal{L}^{(\tilde{\mathcal{M}})} = \tilde{\gamma}(\tilde{\mathcal{M}} - \mathcal{I}). \quad (54)$$

According to the results of the previous section we can then conclude that the continuous trajectory  $\mathcal{T}_t^{(\tilde{\mathcal{M}})}(\rho)$  associated with (53) is again mixing, independently of the ratio  $\lambda$  and of the properties of  $\mathcal{L}^{(\mathcal{E})}$  (the asymptotic convergence point being the fixed point of the ergodic channel  $\tilde{\mathcal{M}}$ ).

#### B. Necessary and Sufficient Condition for the Ergodicity of a Lindblad Generator

A generalization of the result discussed in the previous section to arbitrary Lindblad generators can be obtained by reversing the connection  $\mathcal{M} \rightarrow \mathcal{L}^{(\mathcal{M})}$  of (46). Specifically we will show that the ergodicity of a *generic* Lindblad generator  $\mathcal{L}$  (and hence the asymptotic mixing property of its integrated trajectory  $e^{\mathcal{L}t}$ ) is equivalent to the ergodicity of a suitable quantum channel  $\mathcal{M}_{\mathcal{L}}$  which can be associated to  $\mathcal{L}$ . To see this we recall that any  $\mathcal{L}$  can always be written as

$$\begin{aligned} \mathcal{L}(\rho) &= i[H, \rho] + 2\mathcal{A}(\rho) - \mathcal{A}^\dagger(I)\rho - \rho\mathcal{A}^\dagger(I) \\ &= 2\mathcal{A}(\rho) - G\rho - \rho G^\dagger, \quad G := \mathcal{A}^\dagger(I) - iH, \end{aligned} \quad (55)$$

where  $H = H^\dagger$  is a Hamiltonian operator and  $\mathcal{A}$  is a completely positive (not necessarily trace preserving) map.

Equation (55) can be conveniently rewritten as a difference of two completely positive maps:

$$\mathcal{L} = \mathcal{S}_{\mathcal{L}} - \mathcal{T}_{\mathcal{L}}, \quad (56)$$

$$\mathcal{S}_{\mathcal{L}}(\rho) := 2\mathcal{A}(\rho) + \frac{1}{2}(I - G)\rho(I - G^\dagger), \quad (57)$$

$$\mathcal{T}_{\mathcal{L}}(\rho) := \frac{1}{2}(I + G)\rho(I + G^\dagger). \quad (58)$$

where the second term is invertible with completely positive inverse

$$\mathcal{T}_{\mathcal{L}}^{-1}(\rho) := \frac{1}{2}(I + G)^{-1}\rho(I + G^\dagger)^{-1}, \quad (59)$$

[the invertibility of  $\mathcal{T}_{\mathcal{L}}$  being a direct consequence of the invertibility of the operator  $I + G$ , the latter following from the fact that  $\mathcal{A}^\dagger(I)$  is non negative]. The definition of the quantum channel  $\mathcal{M}_{\mathcal{L}}$  is now obtained by observing that

**Lemma 7.** *The map  $\mathcal{M}_{\mathcal{L}} := \mathcal{S}_{\mathcal{L}} \circ \mathcal{T}_{\mathcal{L}}^{-1}$  is completely positive and trace-preserving.*

*Proof:* Complete positivity is clear. To prove that  $\mathcal{M}_{\mathcal{L}}$  is trace-preserving we show that its dual  $\mathcal{M}_{\mathcal{L}}^\dagger$  is unit-preserving. Indeed, we have

$$\begin{aligned} \mathcal{M}_{\mathcal{L}}^\dagger(I) &= 2(I + G^\dagger)^{-1} \left[ 2\mathcal{A}^\dagger(I) + \frac{1}{2}(I - G^\dagger)(I - G) \right] (I - G)^{-1} \\ &= 2(I + G^\dagger)^{-1} \left[ G + G^\dagger + \frac{1}{2}(I - G^\dagger)(I - G) \right] (I + G)^{-1} \\ &= 2(I + G^\dagger)^{-1} \left[ \frac{1}{2}(I + G^\dagger)(I + G) \right] (I + G)^{-1} = I. \end{aligned} \quad (60)$$

**Theorem 14** (Ergodic semigroups and ergodic channels). *The semigroup generated by  $\mathcal{L}$  in (55) is ergodic if and only if the channel  $\mathcal{M}_{\mathcal{L}}$  is ergodic.*

*Proof:* Let  $A$  be a fixed point of the semigroup generated by  $\mathcal{L}$ , namely  $\mathcal{L}(A) = 0$ . Equivalently, we have  $\mathcal{S}_{\mathcal{L}}(A) = \mathcal{T}_{\mathcal{L}}(A)$  which implies that  $\mathcal{T}_{\mathcal{L}}(A)$  is a fixed point of the channel  $\mathcal{M}_{\mathcal{L}} = \mathcal{S}_{\mathcal{L}} \circ \mathcal{T}_{\mathcal{L}}^{-1}$ . Hence, the semigroup generated by  $\mathcal{L}$  has a unique fixed point if and only if  $\mathcal{M}_{\mathcal{L}}$  has a unique fixed point. ■

As a special instance of the Theorem, we can re-obtain the results of Section VIII-A. Indeed assuming  $\mathcal{L}$  as in (46) we have  $H = 0$ ,  $\mathcal{A} = (\gamma/2)\mathcal{M}$  with  $\mathcal{M}$  being CPTP so that  $G = \mathcal{A}^\dagger(I) = \gamma I/2$ . Hence, the maps  $\mathcal{S}_{\mathcal{L}}$ ,  $\mathcal{T}_{\mathcal{L}}$ , and  $\mathcal{M}_{\mathcal{L}}$  are given by  $\mathcal{S}_{\mathcal{L}} = \gamma\mathcal{M} + \frac{(2-\gamma)^2}{8}\mathcal{I}$ ,  $\mathcal{T}_{\mathcal{L}} = \frac{(2-\gamma)^2}{8}\mathcal{I}$ , and  $\mathcal{M}_{\mathcal{L}} = \frac{8\gamma}{(2+\gamma)^2}\mathcal{M} + \frac{(2-\gamma)^2}{(2+\gamma)^2}\mathcal{I}$ . Note that  $\mathcal{M}_{\mathcal{L}}$  (and hence  $\mathcal{L}$ ) is ergodic if and only if  $\mathcal{M}$  is.

### C. Convexity of Ergodic Semigroups

Consider now two dynamical semigroups, with Lindblad generators  $\mathcal{L}$  and  $\mathcal{L}'$  and define the Lindblad generator

$$\mathcal{L}_p := p\mathcal{L} + (1-p)\mathcal{L}', \quad p \in (0, 1]. \quad (61)$$

Assuming that  $\mathcal{L}$  generates an ergodic semigroup, we may ask whether  $\mathcal{L}_p$  also generates an ergodic semigroup. At the end of Section VIII-A we have already seen that this is indeed the case when (46) hold for both generators. Answering the question for the general case is difficult. Still it is possible to provide a relatively simple answer when, cast in the form (55), the two semigroups have the same Hamiltonian and the same positive operators  $\mathcal{A}^\dagger(I)$  and  $\mathcal{A}'^\dagger(I)$ , that is,

$$H' = H, \quad \mathcal{A}'^\dagger(I) = \mathcal{A}^\dagger(I) \quad (62)$$

(incidentally this case covers also the scenario addressed in Section VIII-A).

**Theorem 15** (Convexity of ergodic semigroups). *Suppose that the condition of (62) is satisfied and that  $\mathcal{L}$  generates an ergodic semigroup. Then,  $\mathcal{L}_p$  generates an ergodic semigroup for every  $p \in (0, 1]$ .*

*Proof:* Under the condition of (62), we have

$$\begin{aligned} \mathcal{S}_{\mathcal{L}_p}(\rho) &= p\mathcal{A}(\rho) + (1-p)\mathcal{A}'(\rho) + \frac{1}{2}(I - G)\rho(I - G^\dagger) \\ &= p\mathcal{S}_{\mathcal{L}}(\rho) + (1-p)\mathcal{S}_{\mathcal{L}'}(\rho), \end{aligned} \quad (63)$$

$$\mathcal{T}_{\mathcal{L}_p}(\rho) = \frac{1}{2}(I + G)\rho(I + G^\dagger) = \mathcal{T}_{\mathcal{L}}(\rho) = \mathcal{T}_{\mathcal{L}'}(\rho), \quad (64)$$

and therefore,  $\mathcal{M}_{\mathcal{L}_p} = p\mathcal{M}_{\mathcal{L}} + (1-p)\mathcal{M}_{\mathcal{L}'}$ . Using Theorems 4 and 14 we then obtain that  $\mathcal{L}_p$  is ergodic. ■

Although this is weaker than the corresponding Theorem 4 for discrete channels, we can still conclude that the set of non-ergodic dynamical semigroups have measure zero. Note that the condition (62) is equivalent to  $G = G'$ . We decompose the set of all Lindblad superoperators into convex sets with  $G = G'$ . Now, each of this set contains at least one ergodic Lindblad superoperator: simply choose  $\mathcal{A}(\rho) = \rho_* \text{Tr}[\rho(G + G^\dagger)/2]$ , where  $\rho_*$  is faithful. The channel  $\mathcal{M}$  associated to this Lindbladian is of the form  $\mathcal{M}(\rho) = \rho_* \text{Tr}[P\rho] + \mathcal{Q}(\rho)$ , where  $P$  is a positive operator and  $\mathcal{Q}$  is a quantum operation. A channel of this form is necessarily ergodic, due to our characterization Theorem 1, because  $\mathcal{M}$  cannot have two distinct invariant subspaces. Therefore, the only non-ergodic maps can be at the boundary of the sets with  $G = G'$ .

## IX. CONCLUSIONS

We have discussed structural properties of quantum channels in finite dimensions, focusing on criteria for ergodicity and mixing. Because these notions are relevant to many protocols in quantum information processing, our characterization paves the way to simpler proofs of quantum convergence in those applications. One of our main results, i.e. the convexity of ergodicity, has potential applications in quantum statistical dynamics, where the ergodicity of a given model is usually hard to establish. Since thermal states provide a natural convex decomposition, our result implies that it suffices to establish ergodicity at zero temperature only. It remains to be seen if a similar result holds for continuous Lindbladian dynamics when the generators that are randomized have different Hamiltonians and/or different positive operators in the dissipative parts of the Lindblad generators.

## APPENDIX

This section is dedicated to the detailed discussions of some of the technical aspects we presented in the main text. We start by presenting a useful Lemma which is often invoked in the text.

**Lemma 8.** *Given  $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$  density matrices, they can be related as*

$$\rho = q\sigma + (1 - q)\tau, \quad (65)$$

with  $q \in (0, 1]$  and  $\tau \in \mathfrak{S}(\mathcal{H})$ , if and only if  $\text{Supp}(\sigma) \subseteq \text{Supp}(\rho)$ .

*Proof:* Consider first the case in which (65) holds for some  $q$  and  $\tau$ . By contradiction assume then that  $\text{Supp}(\rho) \subset \text{Supp}(\sigma)$ , i.e.  $\text{Ker}(\sigma) \subset \text{Ker}(\rho)$ , where  $\text{Ker}(A)$  represents the kernel of a self-adjoint operator  $A$ . This implies that one can identify a vector  $|\psi\rangle$  such that  $|\psi\rangle \in \text{Ker}(\rho)$  and  $|\psi\rangle \notin \text{Ker}(\sigma)$ . For it we must have

$$0 = \langle \psi | \rho | \psi \rangle = q \langle \psi | \sigma | \psi \rangle + (1 - q) \langle \psi | \tau | \psi \rangle > 0, \quad (66)$$

which is clearly absurd, and the necessity is proved. On the contrary, suppose  $\text{Supp}(\sigma) \subseteq \text{Supp}(\rho)$ . Let us take the maximum eigenvalue  $\lambda$  of  $\sigma$  and the minimum non-null eigenvalue  $\mu$  of  $\rho$ . Clearly  $\rho \geq \mu P_\rho$  and  $\sigma \leq \lambda P_\rho$ , where  $P_\rho$  is the projection on the support of  $\rho$ . It is possible to prove that  $\mu \leq \lambda$  since  $\mu \leq 1/r_\rho \leq 1/r_\sigma \leq \lambda$ , with  $r_A$  denoting the rank of a generic operator  $A$ . Now if in the previous relation we have an equality then  $\rho = \sigma = P_\rho/r_\rho$  and the thesis derives immediately. On the other hand, if  $\mu < \lambda$  then we have that  $(\mu/\lambda)\sigma \leq \mu P_\rho \leq \rho$ , which implies  $\rho = (\mu/\lambda)\sigma + (1 - \mu/\lambda)\tau$ , with  $\tau \geq 0$ . But this is exactly of the form (65): indeed since  $\mu < \lambda$  and  $\text{Tr}[\rho] = 1$ , we have  $(1 - \mu/\lambda) \text{Tr}[\tau] = 1 - \mu/\lambda$ , which implies that  $\text{Tr}[\tau] = 1$  and hence  $\tau \in \mathfrak{S}(\mathcal{H})$ . ■

#### A. Alternative Proof of the Convexity of Ergodic Maps

Here we present an alternative derivation of the first part of Theorem 4, namely that convex combinations of the form (29) with  $\mathcal{M}$  ergodic and  $\mathcal{M}'$  generic CPTP maps are also ergodic. Differently from the proof provided in the main text, the approach presented here does not exploit the connection between ergodicity and invariant subspaces. Instead it follows directly from the following result which is a direct consequence of Corollary 2:

**Corollary 9.** *A CPTP map  $\mathcal{M}$  is ergodic, if and only if  $\forall \rho \neq \rho'$  states and  $\forall N, \lambda$  positive constants, we have*

$$\left\| \sum_{n=0}^N \lambda^n \mathcal{M}^n(\rho - \rho') \right\|_1 < f_N(\lambda) \|\rho - \rho'\|_1, \quad (67)$$

where

$$f_N(\lambda) = \sum_{n=0}^N \lambda^n = \begin{cases} N+1 & (\lambda = 1), \\ \frac{1-\lambda^{N+1}}{1-\lambda} & (\lambda \neq 1). \end{cases} \quad (68)$$

*Proof:* If (67) holds then  $\mathcal{M}$  is clearly ergodic. Indeed if this is not the case  $\mathcal{M}$  must have at least two different fixed points  $\rho_* \neq \rho'_*$ , which verify the identity

$$\left\| \sum_{n=0}^N \lambda^n \mathcal{M}^n(\rho_* - \rho'_*) \right\|_1 = f_N(\lambda) \|\rho_* - \rho'_*\|_1, \quad \forall N > 0, \forall \lambda > 0, \quad (69)$$

contradicting the assumption. Conversely, assume  $\mathcal{M} \in \mathfrak{E}_{\mathcal{E}}(\mathcal{H})$ . For any pair of states  $\rho \neq \rho'$  and for any  $N > 0$  and  $\lambda > 0$ , we have

$$\begin{aligned} \left\| \sum_{n=0}^N \lambda^n \mathcal{M}^n(\rho - \rho') \right\|_1 &\leq \sum_{n=0}^N \lambda^n \|\mathcal{M}^n(\rho - \rho')\|_1 \\ &\leq \sum_{n=0}^N \lambda^n \|\rho - \rho'\|_1 \\ &= f_N(\lambda) \|\rho - \rho'\|_1, \end{aligned} \quad (70)$$

where the first inequality follows from the triangular inequality of the trace distance, while the second from the nonexpansiveness of the CPTP maps  $\mathcal{M}^n$ . To conclude the proof we need to show that this upper bound cannot be saturated. The first inequality of (70) can be turned into an identity, if and only if the operators  $\mathcal{M}^n(\rho - \rho')$  ( $n = 0, \dots, N$ ) are all “parallel,” i.e.  $\mathcal{M}^n(\rho - \rho') = \mu_n(\rho - \rho')$  with  $0 \leq \mu_n \leq 1$  ( $n = 0, \dots, N$ ). This implies  $\mu_n = \mu_1^n$  ( $n = 0, \dots, N$ ), so that the second inequality in (70) reduces to

$$\sum_{n=0}^N (\lambda \mu_1)^n \|\rho - \rho'\|_1 \leq \sum_{n=0}^N \lambda^n \|\rho - \rho'\|_1, \quad (71)$$

showing that it can be transformed into an equality if and only if  $\mu_1 = 1$ . Replacing this into the parallelism constraint we can conclude that the upper bound of (70) can be saturated if and only if  $\mathcal{M}(\rho - \rho') = \rho - \rho'$ . However, since  $\mathcal{M}$  is ergodic, from Corollary 2 we must have  $\rho - \rho' = \rho_* \text{Tr}[\rho - \rho'] = 0$ , which contradicts the assumption  $\rho \neq \rho'$ . Therefore, the upper bound of (70), i.e. of (67), cannot be saturated for any pair of states  $\rho \neq \rho'$ . ■

We are now in a position to present our alternative proof that  $\mathcal{M}_p$  of (29) is ergodic.

*Proof of Theorem 4:* Note that for every integer  $n$  one can write  $\mathcal{M}_p^n = p^n \mathcal{M}^n + (1 - p^n) \mathcal{S}_n$ , where  $\mathcal{S}_n$  is a CPTP map. Therefore, for  $N > 0$  and  $\lambda > 0$  we can invoke the triangular inequality of the trace norm to state that

$$\begin{aligned} \left\| \sum_{n=0}^N \lambda^n \mathcal{M}_p^n(\rho - \rho') \right\|_1 &\leq \left\| \sum_{n=0}^N \lambda^n p^n \mathcal{M}^n(\rho - \rho') \right\|_1 \\ &\quad + \left\| \sum_{n=0}^N \lambda^n (1 - p^n) \mathcal{S}_n(\rho - \rho') \right\|_1. \end{aligned} \quad (72)$$

Since  $\mathcal{M}$  is ergodic, we can use Corollary 9 to bound the first term as follows

$$\left\| \sum_{n=0}^N \lambda^n p^n \mathcal{M}^n(\rho - \rho') \right\|_1 < f_N(\lambda p) \|\rho - \rho'\|_1. \quad (73)$$



On the contrary using again the triangular inequality and the nonexpansiveness of  $\mathcal{S}_n$  we have

$$\left\| \sum_{n=0}^N \lambda^n (1-p^n) \mathcal{S}_n(\rho - \rho') \right\|_1 \leq \sum_{n=0}^N \lambda^n (1-p^n) \|\rho - \rho'\|_1 = [f_N(\lambda) - f_N(\lambda p)] \|\rho - \rho'\|_1. \quad (74)$$

Substituting this and (73) in (72) we arrive at

$$\left\| \sum_{n=0}^N \lambda^n \mathcal{M}_p^n(\rho - \rho') \right\|_1 < f_N(\lambda) \|\rho - \rho'\|_1, \quad (75)$$

which according to Corollary 9 is sufficient to claim the ergodicity of  $\mathcal{M}_p$ . ■

### B. Other Proofs

Here we conclude the Appendix by providing the details of the proofs that had been skipped in the main text.

*Proof of Lemma 1:* Since  $A$  is a fixed point, also  $A^\dagger$  is a fixed point [see (8)] and so are the linear combinations  $X := (A + A^\dagger)/2$  and  $Y := (A - A^\dagger)/2i$ . Let us denote by  $P_+$  ( $P_-$ ) the projectors on the eigenspaces of  $X$  with non-negative (negative) eigenvalues and write  $X$  as  $X = X_+ - X_-$  where  $X_+ := P_+ X$  and  $X_- := -P_- X$ . Now, we have  $X_+ = P_+ X = P_+ \mathcal{M}(X) = P_+ \mathcal{M}(X_+) - P_+ \mathcal{M}(X_-)$ , which implies

$$\begin{aligned} \text{Tr}[X_+] &= \text{Tr}[P_+ \mathcal{M}(X_+)] - \text{Tr}[P_+ \mathcal{M}(X_-)] \\ &\leq \text{Tr}[P_+ \mathcal{M}(X_+)] \\ &\leq \text{Tr}[\mathcal{M}(X_+)] \\ &= \text{Tr}[X_+]. \end{aligned} \quad (76)$$

In order for the equality to hold, it is necessary to have  $P_+ \mathcal{M}(X_-) = P_- \mathcal{M}(X_+) = 0$ . Hence, we have  $X_+ = P_+ \mathcal{M}(X_+) = \mathcal{M}(X_+)$ , which also implies  $X_- = \mathcal{M}(X_-)$ . Repeating the same reasoning for  $Y = Y_+ - Y_-$  we obtain that also  $Y_+$  and  $Y_-$  are fixed points of  $\mathcal{M}$ . ■

*Alternative Proof of Corollary 2:* From Lemma 6 of [9], we know that  $|A| := \sqrt{A^\dagger A}$  must be a fixed point state of the map. If  $\mathcal{M}$  is ergodic with fixed point state  $\rho_*$  then we must have  $|A| = \|A\|_1 \rho_*$ . Consider first the case in which  $A$  is Hermitian, i.e.  $A = A^\dagger$  (the non-Hermitian case will be considered below). In this case we can then write  $\rho_* = \sum_n |c_n| |\phi_n\rangle\langle\phi_n| / \sum_m |c_m|$ , with  $\{|\phi_n\rangle\}$  being the orthonormal eigenvectors of  $A$  and  $c_n$  the corresponding eigenvalues. Furthermore for each  $\alpha$  real and satisfying the inequality  $0 < |\alpha| \leq 1 / \sum_m |c_m|$ , we can also conclude that the operator

$$\begin{aligned} \tilde{\rho}_* &= \frac{\alpha A + \rho_*}{\alpha \text{Tr}[A] + 1} \\ &= \frac{1}{\alpha \sum_m c_m + 1} \sum_n \left( \alpha c_n + \frac{|c_n|}{\sum_m |c_m|} \right) |\phi_n\rangle\langle\phi_n| \end{aligned} \quad (77)$$

is a density matrix of the system (indeed it has trace 1 and is positive semidefinite) which by construction is also a fixed point of  $\mathcal{M}$ . Therefore, since  $\mathcal{M}$  is ergodic, we must have  $\tilde{\rho}_* = \rho_*$  or  $A = \rho_* \text{Tr}[A]$ , as required by the Corollary 2.

Suppose now that  $A$  is not Hermitian. Since  $A^\dagger$  is also an eigenoperator with eigenvalue 1, the Hermitian operators  $(A + A^\dagger)/2$  and  $(A - A^\dagger)/2i$  are also solutions of (15). Hence, from the previous derivation, we must have  $(A + A^\dagger)/2 = \rho_* \text{Tr}[A + A^\dagger]/2$  and  $(A - A^\dagger)/2i = \rho_* \text{Tr}[A - A^\dagger]/2i$ , which yields (16). ■

*Proof of Theorem 11:* If (42) holds, then we have  $\mathcal{M}(A) = \sum_i M_i A M_i^\dagger = \omega A \sum_i M_i M_i^\dagger = \omega A \mathcal{M}(I) = \omega A$ . Hence,  $A$  is an eigenvector of  $\mathcal{M}$  with eigenvalue  $\omega$ . Conversely, suppose that  $A$  is eigenvector belonging to eigenvalue  $\omega$  with  $|\omega| = 1$ . Equivalently, we have  $\mathcal{M}^\dagger(A) = \bar{\omega} A$  (cf. Corollary 6). Writing  $A$  in the polar decomposition  $A = U|A|$  and choosing a Kraus form for  $\mathcal{M}$  we have

$$|A| = \omega \sum_i (U^\dagger M_i^\dagger U) |A| M_i. \quad (78)$$

Let us diagonalize  $|A|$  as  $|A| = \sum_{k=1}^s \alpha_k P_k$ , where  $\alpha_1 > \alpha_2 > \dots > \alpha_s \geq 0$  are the eigenvalues,  $S_k$  is the eigenspace corresponding to  $\alpha_k$ , and  $P_k$  is the projector on  $S_k$ . Then, for every unit vector  $\varphi \in S_1$  we have

$$\begin{aligned} \alpha_1 &= \langle \varphi | |A| | \varphi \rangle \\ &= \sum_i \omega \langle \varphi | U^\dagger M_i^\dagger U | A | M_i | \varphi \rangle \\ &\leq \sqrt{\sum_i \langle \varphi | U^\dagger M_i^\dagger U | A | U^\dagger M_i U | \varphi \rangle \cdot \sum_j \langle \varphi | M_j^\dagger | A | M_j | \varphi \rangle}, \end{aligned} \quad (79)$$

where the inequality comes from the Cauchy-Schwarz inequality. To saturate the inequality (79) we need to have

$$\exists \lambda \geq 0 \text{ s.t. } |A|^{1/2} U^\dagger M_i U | \varphi \rangle = \lambda \omega |A|^{1/2} M_i | \varphi \rangle, \quad \forall i. \quad (80)$$

Continuing the inequality (79) we then get

$$\begin{aligned} \alpha_1 &\leq \alpha_1 \sqrt{\sum_i \langle \varphi | (U^\dagger M_i^\dagger U) (U^\dagger M_i U) | \varphi \rangle \cdot \sum_j \langle \varphi | M_j^\dagger M_j | \varphi \rangle} \\ &= \alpha_1. \end{aligned} \quad (81)$$

Therefore, both inequalities (79) and (81) must be saturated. To saturate the inequality (81) it is necessary to have  $M_i | \varphi \rangle \in S_1$  and  $U^\dagger M_i U | \varphi \rangle \in S_1$  for every  $i$ , so that (80) becomes

$$\exists \lambda \geq 0 \text{ s.t. } U^\dagger M_i U | \varphi \rangle = \lambda \omega M_i | \varphi \rangle, \quad \forall i. \quad (82)$$

Clearly, in order for  $\mathcal{M}$  to be trace-preserving we must have  $\lambda = 1$ , as  $\lambda^2 = \sum_i \langle \varphi | (\lambda \bar{\omega} M_i^\dagger) (\lambda \omega M_i) | \varphi \rangle = \sum_i \langle \varphi | (U^\dagger M_i^\dagger U) (U^\dagger M_i U) | \varphi \rangle = 1$ . Moreover, since  $| \varphi \rangle$  is a generic element of  $S_1$ , (82) with  $\lambda = 1$  is equivalent to

$$(U^\dagger M_i U) P_1 = \omega M_i P_1, \quad \forall i. \quad (83)$$

Similarly, the relation  $M_i | \varphi \rangle \in S_1, \forall | \varphi \rangle \in S_1$ , which was needed to saturate the inequality (81), is equivalent to

$$M_i P_1 = P_1 M_i P_1, \quad \forall i. \quad (84)$$

Recalling that the eigenvalue equation  $\mathcal{M}^\dagger(A) = \bar{\omega} A$  is equivalent to  $\mathcal{M}(A) = \omega A$  (Corollary 6), we can use the same reasoning as above to prove also

$$M_i^\dagger P_1 = P_1 M_i^\dagger P_1, \quad \forall i. \quad (85)$$

Putting together the two relations (84) and (85) we then obtain  $M_i P_1 = P_1 M_i$ ,  $\forall i$ . Finally, defining the partial isometry  $T_1 := UP_1$  we obtain

$$\begin{aligned} M_i T_1 &= M_i U P_1 = U (U^\dagger M_i U) P_1 \\ &= U (\omega M_i P_1) = \omega (U P_1) M_i = \omega T_1 M_i, \quad \forall i. \end{aligned} \quad (86)$$

Hence, we proved that the partial isometry  $T_1$  must satisfy (42) in the statement of the theorem. In particular, we then have  $\mathcal{M}(T_1) = \omega T_1$ . This means that for every peripheral eigenvalue  $\omega$ , the channel  $\mathcal{M}$  must have at least one eigenvector that is a partial isometry and satisfies (42). Moreover, defining the operator  $A' := A - \alpha_1 T_1$  we have  $\mathcal{M}(A') = \omega A'$ . The polar decomposition of  $A'$  is  $A' = U(|A| - \alpha_1 P_1) = U \sum_{k=2}^s \alpha_k P_k$ , so that the eigenspace of  $|A'|$  with maximum eigenvalue is  $S_2$ . Iterating the above proof we obtain that the partial isometry  $T_2 := UP_2$  is an eigenvector of  $\mathcal{M}$  satisfying (42), and by further iteration we obtain that every partial isometry  $T_k := UP_k$  is an eigenvector of  $\mathcal{M}$  satisfying (42). In conclusion, the operator  $A$  is a linear combination of partial isometries satisfying (42) and, by linearity, it satisfies (42). ■

*Proof of Proposition 2:* Suppose that  $\mathcal{M}^\dagger \mathcal{M}$  is not ergodic. Hence, by Theorem 12 there exists a projector  $P < I$  such that  $(\mathcal{M}^\dagger \mathcal{M})(P) = P$ . Using this fact we obtain

$$\langle \mathcal{M}(I - P), \mathcal{M}(P) \rangle = \langle I - P, \mathcal{M}^\dagger \mathcal{M}(P) \rangle = \langle I - P, P \rangle = 0. \quad (87)$$

Since  $\text{Tr}[\mathcal{M}(I - P)\mathcal{M}(P)] = 0$ , we necessarily have  $\mathcal{M}(I - P)\mathcal{M}(P) = 0$ . In addition we have  $\mathcal{M}(I - P) + \mathcal{M}(P) = \mathcal{M}(I) = I$ . It is then easy to see that  $\mathcal{M}(P)$  is an orthogonal projector: we have

$$\begin{aligned} [\mathcal{M}(P)]^2 &= \mathcal{M}(P)[\mathcal{M}(P) + \mathcal{M}(I - P)] \\ &= \mathcal{M}(P)\mathcal{M}(I) = \mathcal{M}(P). \end{aligned} \quad (88)$$

Since we have  $\text{Tr}[\mathcal{M}(P)] = \text{Tr}[P]$ , the dimensions of the support of  $\mathcal{M}(P)$  and of  $P$  coincide. Hence, there exists a unitary  $U$  such that  $\mathcal{M}(P) = UPU^\dagger$ . Conversely, suppose that the relation  $\mathcal{M}(P) = UPU^\dagger$  is satisfied for some projector  $P < I$  and for some unitary  $U$ . Then, we have

$$\begin{aligned} \text{Tr}[P] &= \langle UPU^\dagger, \mathcal{M}(P) \rangle \\ &= \langle \mathcal{M}^\dagger(UPU^\dagger), P \rangle \\ &\leq \sqrt{\langle \mathcal{M}^\dagger(UPU^\dagger), \mathcal{M}^\dagger(UPU^\dagger) \rangle \langle P, P \rangle} \\ &= \sqrt{\text{Tr}[UPU^\dagger \mathcal{M} \mathcal{M}^\dagger(UPU^\dagger)] \text{Tr}[P]} \\ &\leq \sqrt{\text{Tr}[\mathcal{M} \mathcal{M}^\dagger(UPU^\dagger)] \text{Tr}[P]} = \text{Tr}[P], \end{aligned} \quad (89)$$

which requires  $\mathcal{M}^\dagger(UPU^\dagger) = P$ . Hence, we have  $\mathcal{M}^\dagger \mathcal{M}(P) = \mathcal{M}^\dagger(UPU^\dagger) = P$ , namely  $P$  is a fixed point of  $\mathcal{M}^\dagger \mathcal{M}$ . Hence,  $\mathcal{M}^\dagger \mathcal{M}$  is not ergodic. ■

## REFERENCES

- [1] R. F. Streater, *Statistical Dynamics*, 2nd ed. London: Imperial College Press, 2009.
- [2] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*. Oxford: Oxford University Press, 2002.
- [3] F. Benatti and R. Floreanini, Eds., *Irreversible Quantum Dynamics*. Berlin: Springer, 2003.
- [4] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, 2nd ed. Berlin: Springer, 2007.
- [5] A. S. Holevo and V. Giovannetti, "Quantum channels and their entropic characteristics," *Rep. Prog. Phys.*, vol. 75, no. 4, p. 046001, Mar. 2012.
- [6] M. Keyl, "Fundamentals of quantum information theory," *Phys. Rep.*, vol. 369, no. 5, pp. 431–548, Oct. 2002.
- [7] C. Bennett and P. Shor, "Quantum information theory," *IEEE Trans. Inf. Theory*, vol. 44, no. 6, pp. 2724–2742, Oct. 1998.
- [8] M. M. Wolf, "Quantum channels & operations: Guided tour," unpublished.
- [9] D. Burgarth and V. Giovannetti, "The generalized Lyapunov theorem and its application to quantum channels," *New J. Phys.*, vol. 9, no. 5, p. 150, May 2007.
- [10] M. Raginsky, "Dynamical aspects of information storage in quantum-mechanical systems," Ph.D. dissertation, Department of Electrical and Computer Engineering, Northwestern University, Chicago, Jun. 2002.
- [11] M. Raginsky, "Strictly contractive quantum channels and physically realizable quantum computers," *Phys. Rev. A*, vol. 65, no. 3, p. 032306, Feb. 2002.
- [12] T. Wellens, A. Buchleitner, B. Kümmerer, and H. Maassen, "Quantum state preparation via asymptotic completeness," *Phys. Rev. Lett.*, vol. 85, no. 16, pp. 3361–3364, Oct. 2000.
- [13] D. Burgarth and V. Giovannetti, "Mediated homogenization," *Phys. Rev. A*, vol. 76, no. 6, p. 062307, Dec. 2007.
- [14] D. Burgarth and V. Giovannetti, "A protocol for cooling and controlling composite systems by local interactions," in *Quantum Information and Many Body Quantum Systems: Proceedings*, M. Ericsson and S. Montangero, Eds. Pisa: Edizioni della Normale, Dec. 2008, p. 17.
- [15] D. Burgarth and V. Giovannetti, "Full control by locally induced relaxation," *Phys. Rev. Lett.*, vol. 99, no. 10, p. 100501, Sep. 2007.
- [16] D. Burgarth, S. Bose, C. Bruder, and V. Giovannetti, "Local controllability of quantum networks," *Phys. Rev. A*, vol. 79, no. 6, p. 060305(R), Jun. 2009.
- [17] K. Yuasa, D. Burgarth, V. Giovannetti, and H. Nakazato, "Efficient generation of a maximally entangled state by repeated on- and off-resonant scattering of ancilla qubits," *New J. Phys.*, vol. 11, no. 12, p. 123027, Dec. 2009.
- [18] M. Guță, "Fisher information and asymptotic normality in system identification for quantum Markov chains," *Phys. Rev. A*, vol. 83, no. 6, p. 062324, Jun. 2011.
- [19] V. Giovannetti and D. Burgarth, "Improved transfer of quantum information using a local memory," *Phys. Rev. Lett.*, vol. 96, no. 3, p. 030501, Jan. 2006.
- [20] D. Burgarth, V. Giovannetti, and S. Bose, "Optimal quantum-chain communication by end gates," *Phys. Rev. A*, vol. 75, no. 6, p. 062327, Jun. 2007.
- [21] M. Fannes, B. Nachtergaele, and R. F. Werner, "Finitely correlated states on quantum spin chains," *Commun. Math. Phys.*, vol. 144, no. 3, pp. 443–490, 1992.
- [22] V. Giovannetti, S. Montangero, and R. Fazio, "Quantum multiscale entanglement renormalization ansatz channels," *Phys. Rev. Lett.*, vol. 101, no. 18, p. 180503, Oct. 2008.
- [23] V. Giovannetti, S. Montangero, M. Rizzi, and R. Fazio, "Homogeneous multiscale-entanglement-renormalization-ansatz states: An information theoretical analysis," *Phys. Rev. A*, vol. 79, no. 5, p. 052314, May 2009.
- [24] M. Rizzi, S. Montangero, P. Silvi, V. Giovannetti, and R. Fazio, "Homogeneous multiscale entanglement renormalization ansatz tensor networks for quantum critical systems," *New J. Phys.*, vol. 12, no. 7, p. 075018, Jul. 2010.
- [25] P. Silvi, V. Giovannetti, P. Calabrese, G. E. Santoro, and R. Fazio, "Entanglement renormalization and boundary critical phenomena," *J. Stat. Mech.*, p. L03001, Mar. 2010.
- [26] P. Silvi, V. Giovannetti, S. Montangero, M. Rizzi, J. I. Cirac, and R. Fazio, "Homogeneous binary trees as ground states of quantum critical hamiltonians," *Phys. Rev. A*, vol. 81, no. 6, p. 062335, Jun. 2010.
- [27] F. Haag, "Asymptotisches Verhalten von quanten-Markov-halbgruppen und quanten-Markov-prozessen," Ph.D. dissertation, TU Darmstadt, Aachen, Jan. 2006.
- [28] M. Wolf and J. Cirac, "Dividing quantum channels," *Commun. Math. Phys.*, vol. 279, no. 1, pp. 147–168, 2008.
- [29] L. J. Landau and R. F. Streater, "On Birkhoff's theorem for doubly stochastic completely positive maps of matrix algebras," *Linear Algebra Appl.*, vol. 193, pp. 107–127, Nov. 1993.

- [30] M. Sanz, D. Pérez-García, M. M. Wolf, and J. I. Cirac, "A quantum version of Wielandt's inequality," *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4668–4673, Sep. 2010.
- [31] S. L. Tregub, "Bistochastic operators on finite-dimensional von Neumann algebras," *Soviet Math.*, vol. 30, no. 3, pp. 105–108, 1986.
- [32] B. Kümmerer and H. Maassen, "The essentially commutative dilations of dynamical semigroups on  $m_n$ ," *Commun. Math. Phys.*, vol. 109, no. 1, pp. 1–22, 1987.
- [33] S. G. Schirmer and X. Wang, "Stabilizing open quantum systems by Markovian reservoir engineering," *Phys. Rev. A*, vol. 81, no. 6, p. 062306, Jun. 2010.
- [34] B. Baumgartner and H. Narnhofer, "The structures of state space concerning quantum dynamical semigroups," *Rev. Math. Phys.*, vol. 24, no. 2, p. 1250001, Mar. 2012.